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Gaussian Approximation to the Condensation of an Interacting Bose Gas

A. Okopińska

Institute of Physics, Pedagogical University, Świętokrzyska 15, Kielce, 25-406 Poland

e-mail: okopin@fuw.edu.pl

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Abstract—The effective action formalism of quantum-field theory is used to study the properties of a nonrelativistic interacting Bose gas. The Gaussian approximation is formulated by calculating the effective action to the first order of the optimized expansion. In the homogeneous limit, the method respects the Hughenholz–Pines theorem, leading to a gapless spectrum both for excitations and for density fluctuations. Renormalization is carried out by adopting dimensional regularization. The results for critical temperature are compared with those obtained in the loop expansion and lattice calculations.

1. INTRODUCTION

A weakly interacting Bose gas was intensively studied in the period between 1947 and 1965 as a theoretical example of a Bose-condensed system described by non-relativistic QFT. Various approximation methods have been considered [1–5], and great success has been achieved in describing the essential features of ^{4}He superfluidity. However, the interactions in quantum liquids are too strong for a quantitative test of many-body methods. The weakly interacting system became accessible only in 1995 after the experimental verification of Bose–Einstein condensation (BEC) in dilute gases in magnetic traps [5]. When adapting many-body approximations to trapped gases, the problem of approximation consistency [6] attracted renewed attention [7]. The approximation scheme should preserve relations between physical observables arising from the symmetry of the theory (Ward identities). For spatially uniform systems, such a relation is implied by the Hughenholz–Pines (H–P) theorem [8] (a nonrelativistic analogue of the Goldstone theorem). The H–P theorem shows that the single-particle spectrum of a many-body system is gapless if the global symmetry is spontaneously broken. This imposes strong constraints on approximations. The simplest Bogoliubov approximation [1] fulfils these constraints, but inclusion of the first-order corrections to propagators violates the H–P theorem. Only the second-order approximation, which was developed by Beliaev [3] and extended to finite temperature by Popov, is fully consistent. In the self-consistent Hartree–Fock–Bogoliubov approximation, the H–P is violated: there is a gap in the single-particle spectrum even though the density fluctuations are gapless [7].

The Bose-condensed systems are usually studied using field theoretical techniques involving the particle Hamiltonian. The Lagrangian formulation is more convenient for discussing general issues, such as symmetries, conservation laws, and Ward identities. Upon straightforward extension to finite temperature, the grand canonical thermodynamic potential is obtained as a function of the superfluid order parameter. However, the results obtained in the Lagrangian approach are also controversial; Toyoda [9] claimed the one-loop effective potential at finite temperature to be consistent with the H–P theorem, but later the opposite was stated [10]. Methods of self-consistent resummation have also been discussed [10, 11], but the consistency with the H–P theorem is unclear. We show that the effective action provides a useful tool to construct consistent approximations.

The outline of our work is as follows. In Section 2, the effective action formalism is introduced using path-integral representation of generating functionals in nonrelativistic QFT. We show that the H–P theorem can be formulated in the form of a simple criterium, which is useful for checking the consistency of approximations. In Section 3, we study the loop expansion for the effective action. The one-loop result gives the Beliaev–Popov approximation at finite temperature. In Section 4, we extend the method of the optimized expansion [12] to the effective action of nonrelativistic theory. The self-consistent Gaussian approximation, obtained in the first order, fulfils the H–P theorem. Results for the critical temperature and conclusions are summarized in Section 5.

2. EFFECTIVE ACTION

The system of many spinless atoms interacting by two-body forces is described by a complex scalar field $\Phi(\mathbf{r}, t)$ with the Lagrangian density

$$L[\Phi] = \frac{i}{2}\left(\Phi^*(\mathbf{r},t)\frac{\partial\Phi(\mathbf{r},t)}{\partial t} - \frac{\partial\Phi^*(\mathbf{r},t)}{\partial t}\Phi(\mathbf{r},t)\right) - \frac{\hbar^2}{2m}\nabla\Phi^*(\mathbf{r},t)\cdot\nabla\Phi(\mathbf{r},t) - V_{\text{ext}}(\mathbf{r})\Phi^*(\mathbf{r},t)\Phi(\mathbf{r},t) - \frac{1}{2}\int d^3r'(\Phi^*(\mathbf{r},t)\Phi(\mathbf{r},t)U(\mathbf{r}-\mathbf{r}')\Phi^*(\mathbf{r}',t)\Phi(\mathbf{r}',t)), \quad (1)$$

where $U(\mathbf{r} - \mathbf{r}')$ represents the two-body interatomic potential and $V_{\text{ext}}(\mathbf{r})$ the external potential of the trap. The Lagrangian is invariant under the global U(1) transformation

$$(\Phi, \Phi^*) \longrightarrow (e^{i\alpha}\Phi, e^{-i\alpha}\Phi^*), \quad (2)$$

where α is a constant phase. Since BEC takes place at very low energies, the interatomic potential can be approximated by the local potential $U(\mathbf{r} - \mathbf{r}') = 2\lambda\delta(\mathbf{r} - \mathbf{r}')$, with $\lambda = \frac{2\pi\hbar^2 a}{m}$ related to the scattering length a. The Lagrangian simplifies to the form

$$L[\Phi] = \frac{i}{2}\left(\Phi^*(\mathbf{r},t)\frac{\partial\Phi(\mathbf{r},t)}{\partial t} - \frac{\partial\Phi^*(\mathbf{r},t)}{\partial t}\Phi(\mathbf{r},t)\right) - \frac{\hbar^2}{2m}\nabla\Phi^*(\mathbf{r},t)\cdot\nabla\Phi(\mathbf{r},t) - V_{\text{ext}}(\mathbf{r})\Phi^*(\mathbf{r},t)\Phi(\mathbf{r},t) - \lambda(\Phi^*(\mathbf{r},t)\Phi(\mathbf{r},t))^2. \quad (3)$$

The grand canonical system at temperature T in the imaginary time formalism is defined by Wick's rotated Lagrangian:

$$L_\mu[\Phi] = \Phi^*(\mathbf{r},\tau)\left(-\frac{\partial}{\partial\tau} + \frac{\hbar^2}{2m}\nabla^2 - V_{\text{ext}}(\mathbf{r}) + \mu\right)\Phi(\mathbf{r},\tau) - \lambda(\Phi^*(\mathbf{r},\tau)\Phi(\mathbf{r},\tau))^2, \quad (4)$$

where the chemical potential μ is introduced in order to consider states with an indefinite number of particles. Its value is adjusted such that the expectation value of the number operator is equal to N, corresponding to a fixed particle density $n = \frac{N}{\int d^3r}$. The Euclidean generating functional is defined as a path integral,

$$Z[J] = \int D\Phi D\Phi^* \times e^{-\frac{1}{\hbar}\int_0^\beta d\tau\int d^3r[L_\mu[\Phi] + J^*(\mathbf{r},\tau)\Phi(\mathbf{r},\tau) + J(\mathbf{r},\tau)\Phi^*(\mathbf{r},\tau)]}, \quad (5)$$

over the functions $\Phi(\mathbf{r}, \tau)$ with a period $\beta = \frac{\hbar}{k_B T}$ in τ. The partition function at thermal equilibrium is given by $Z[0]$. The generating functional for the connected Green's functions is $W[J] = \ln Z[J]$. The effective action functional is given by the Legendre transform:

$$\Gamma[\Phi] = W[J] - \int dx J(x)\Phi^*(x) - \int dx J^*(x)\Phi(x), \quad (6)$$

where $x = (\tau, \mathbf{r})$. The background field $\Phi(x) = \frac{\delta W}{\delta J(x)} = \langle\hat{\Phi}(x)\rangle_J$ is a vacuum expectation of the quantum-field operator in the presence of an external source, $J(x) = \frac{1}{\sqrt{2}}(j_1(x) + ij_2(x))$. In performing the calculation, we will use the two real components of the complex field $\Phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$ as independent variables. As a Legendre transform, the effective action fulfils

$$\frac{\delta\Gamma}{\delta\phi_i(x)} = -j_i(x), \quad \text{where} \quad i = 1, 2. \quad (7)$$

The physical value of the background field $\Phi^{(0)}(x)$, corresponding to $J(x) = 0$, is thus determined by the stationarity equations

$$\left.\frac{\delta\Gamma}{\delta\phi_i(x)}\right|_{\Phi^{(0)}(x)} = 0, \quad \text{where} \quad i = 1, 2. \quad (8)$$

The Green's functions are obtained by functional differentiation of the generating functionals. The one-particle Green's function (propagator) matrix reads

$$G_{ij}(x,y) = \left.\frac{1}{Z[0]}\frac{\delta^2 Z}{\delta j_i(x)\delta j_j(y)}\right|_{J=0} = \left.\frac{\delta^2 W}{\delta j_i(x)\delta j_j(y)}\right|_{J=0}.$$

By differentiation of the effective action, the one-particle irreducible Green's functions (proper vertices) are generated. The proper vertex

$$\Gamma_{ij}(x,y) = \left.\frac{\delta^2\Gamma}{\delta\phi_i(x)\phi_j(y)}\right|_{\Phi=\Phi^{(0)}} = \left[\left.\frac{\delta^2 W}{\delta j_i(x) j_j(y)}\right|_{J=0}\right]^{-1} = G^{-1}_{ij}(x,y). \quad (9)$$

One-particle excitations, related to the poles of the full propagator, are determined by zero modes of the Fou-

rier transform $\Gamma_{ij}(p) = \int dx e^{-ip(x-y)} \frac{\delta^2\Gamma}{\delta\phi_i(x)\phi_j(y)}\Big|_{\Phi=\Phi^{(0)}}$, where p stands for $(\omega, \mathbf{p})$. The matrix can be written in the form

$$\Gamma(p) = \begin{pmatrix} \frac{p^2}{2m} - \mu + \Pi_{11} & \omega + \Pi_{12} \\ -\omega + \Pi_{21} & \frac{p^2}{2m} - \mu + \Pi_{22} \end{pmatrix},$$

where $\Pi_{11} = \Sigma_{11} + \frac{1}{2}(\Sigma_{12} + \Sigma_{12}^*)$, $\Pi_{22} = \Sigma_{11} - \frac{1}{2}(\Sigma_{12} + \Sigma_{12}^*)$, and $\Pi_{12} = \frac{i}{2}(\Sigma_{12} - \Sigma_{12}^*)$, with $\Sigma_{11} = \Sigma_{22}$ being the normal self-energy and $\Sigma_{12} = \Sigma_{21}^*$, the anomalous energy.

In the following, we study a homogeneous system $V_{\text{ext}}(\mathbf{r}) \longrightarrow 0$ when the background field $\Phi^{(0)}(x) = \Phi^{(0)}$. In this case, the effective potential

$$V(\Phi) = -\frac{\Gamma[\Phi]|_{\Phi(x)=\Phi=\text{const}}}{\beta\int d^3r} \tag{10}$$

is a useful tool, since $\Phi^{(0)}$ can be determined by the stationarity equation

$$\frac{dV}{d\phi_1}\Big|_{\Phi^{(0)}} = \frac{dV}{d\phi_2}\Big|_{\Phi^{(0)}} = 0. \tag{11}$$

The H–P theorem can be easily demonstrated in the effective action approach. Because of the invariance of $L_\mu[\phi_1, \phi_2]$ under transformation (2), whose infinitesimal version is given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{12}$$

the generating functional $W[j_1, j_2]$ is invariant under

$$\begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix}\begin{pmatrix} j_1 \\ j_2 \end{pmatrix}. \tag{13}$$

This implies

$$\int dx\left(\frac{\delta W}{\delta j_1}j_2 - \frac{\delta W}{\delta j_2}j_1\right) = \int dx\left(-\phi_1\frac{\delta\Gamma}{\delta\phi_2} + \phi_2\frac{\delta\Gamma}{\delta\phi_1}\right) = 0. \tag{14}$$

Taking the derivative over $\phi_2(y)$ at $\Phi^{(0)} = \frac{1}{\sqrt{2}}(\phi_1^{(0)}, 0)$, which fulfils the symmetry breaking condition $\frac{\delta\Gamma}{\delta\phi_1(x)}\Big|_{\Phi^{(0)}} = 0$, we obtain

$$\int dx\left(-\phi_1(x)\frac{\delta^2\Gamma}{\delta\phi_2(x)\phi_2(y)}\Big|_{\Phi^{(0)}} + \frac{\delta\Gamma}{\delta\phi_1(x)}\Big|_{\Phi^{(0)}} + \phi_2(x)\frac{\delta^2\Gamma}{\delta\phi_1(x)\delta\phi_2(y)}\Big|_{\Phi^{(0)}}\right) \tag{15}$$

$$= -\phi_1^{(0)}\int dx\frac{\delta^2\Gamma}{\delta\phi_2(x)\phi_2(y)}\Big|_{\Phi^{(0)}} = -\phi_1^{(0)}\Gamma^{22}(p=0) = 0.$$

Since $\phi_1^{(0)} \neq 0$, this means that $\Gamma^{22}(\omega = 0, \vec{k} = 0) = 0$; therefore, there is a zero-frequency excitation. The H–P theorem can be expressed in the form

$$\Gamma^{22}(\omega = 0, \vec{k} = 0) = \frac{d^2V}{d\phi_2^2}\Big|_{\Phi^{(0)}} = 0. \tag{16}$$

It is easy to observe that the gapless spectrum is just a consequence of the fact that the effective potential is a function of $|\Phi|^2$ and does not depend on ϕ_1 and ϕ_2 separately. This provides a useful criterion for the consistency of an approximation with the H–P theorem.

For interacting atoms ($a \neq 0$), the effective cannot be calculated exactly, so one has to resort to approximations. The advantage of formulating the approximation for the effective action lies in the fact that all the Green's functions are obtained in a consistent way, through functional differentiation.

3. LOOP EXPANSION

The loop expansion is generated by calculating the path integral for $Z[J]$ by the steepest descent method. Upon Legendre transformation, the effective action is obtained as a series in $\hbar$, whose power indicates the number of loops. The 1-loop effective action

$$\Gamma^{1\text{-loop}}[\Phi] = \int dx\left[\frac{i}{2}\phi_1\frac{\partial\phi_2}{\partial\tau} - \frac{i}{2}\phi_2\frac{\partial\phi_1}{\partial\tau} - \frac{1}{2}\phi_1\left(\frac{\nabla^2}{2m} + \mu\right)\phi_1\right. \tag{17}$$

$$\left. - \frac{1}{2}\phi_2\left(\frac{\nabla^2}{2m} + \mu\right)\phi_2 + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2\right] + \frac{\hbar}{2}\text{Tr}\,LnM[\Phi],$$

where

$$M[\Phi] = \begin{bmatrix} -\frac{\nabla^2}{2m} - \mu + 3\lambda\phi_1^2(x) + \lambda\phi_2^2(x) & i\frac{\partial}{\partial\tau} + 2\lambda\phi_1(x)\phi_2(x) \\ -i\frac{\partial}{\partial\tau} + 2\lambda\phi_1(x)\phi_2(x) & -\frac{\nabla^2}{2m} - \mu + \lambda\phi_1^2(x) + 3\lambda\phi_2^2(x) \end{bmatrix},$$

gives the Beliaev–Popov approximation at finite temperature. Setting $\Phi(x) = \Phi$ yields Toyoda's [9] result for a 1-loop finite temperature effective potential:

$$V^{1\text{-loop}}(|\Phi|^2) = -\mu|\Phi|^2 + \lambda|\Phi|^4 + \hbar I_1(\mu, |\Phi|^2), \quad (18)$$

where

$$I_1(\mu, |\Phi|^2) = \int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}\omega_k + \frac{1}{\beta}\ln(1 - e^{-\beta\omega_k})\right] \quad (19)$$

and $\omega_k = \sqrt{\left(\frac{k^2}{2m} - \mu + 4\lambda|\Phi|^2\right)^2 - 4\lambda^2|\Phi|^4}$. The H–P theorem is respected in the 1-loop approximation, since the effective potential depends only on $|\phi|^2$. The incorrect statement in [10] was due to a misinterpretation of the excitation energy. One can explicitly show that $\Gamma_{22}(p = 0) = \left.\frac{d^2V^{1\text{-loop}}}{d\phi_2^2}\right|_{\Phi^{(0)}} = 0$ if $\Phi^{(0)} = \frac{1}{\sqrt{2}}(\phi_1^{(0)}, 0)$ is a solution to the stationarity equation

$$\left.\frac{dV^{1\text{-loop}}}{d\phi_1}\right|_{\Phi^{(0)}} = [-\mu + 2\lambda|\Phi^{(0)}|^2 + \hbar\lambda J_0(\mu, |\Phi^{(0)}|^2)]\phi_1^{(0)} = 0, \quad (20)$$

where

$$J_0(\mu, |\Phi|^2) = 2\int\frac{d^3k}{(2\pi)^3}\frac{\frac{k^2}{2m} - \mu + 3\lambda|\Phi|^2}{\omega_k}(1 + 2n_B(\omega_k)) \quad (21)$$

and the Bose–Einstein distribution function $n_B(\omega) = \frac{1}{(e^{\beta\omega} - 1)}$.

The chemical potential μ can be eliminated in favor of the particle number density using the relation

$$n = -\left.\frac{dV}{d\mu}\right|_{\Phi^{(0)}} = |\Phi_0|^2 + \hbar I_0(\mu, |\Phi_0|^2), \quad (22)$$

where

$$I_0(\mu, |\Phi|^2) = \int\frac{d^3k}{(2\pi)^3}\frac{\frac{k^2}{2m} - \mu + 4\lambda|\Phi|^2}{2\omega_k}(1 + 2n_B(\omega_k)). \quad (23)$$

Renormalization can be carried out by dimensional regularization. Denoting the particle density in condensed state $|\Phi_0|^2$ by n_0 and observing that in the leading order $n_0 = n$, the particle density to the order $\hbar$ can be written as

$$n = n_0 + \int\frac{d^3k}{(2\pi)^3}\frac{\frac{k^2}{2m} + 2\lambda n}{2\omega_k^{\text{Bog}}}(1 + 2n_B(\omega_k^{\text{Bog}})), \quad (24)$$

where the Bogoliubov frequency $\omega_k^{\text{Bog}} = \sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m} + 4\lambda n\right)}$. This expression has been used [13] to calculate condensate depletion:

$$n_0 = n - \frac{8(an)^{3/2}}{3\sqrt{\pi}} - \left(\frac{m}{2\pi\beta}\right)^{2/3}\left[\zeta(3/2) - 2\pi\sqrt{\frac{2\beta an}{m}} + O(\beta)\right]. \quad (25)$$

The depletion and other physical quantities (energy density, pressure, etc.) derived from the 1-loop effective potential are in agreement with the results obtained by Lee and Yang [2] to the lowest order in $an^{1/3}$. However, the 1-loop results are dubious in the vicinity of the phase transition, since the higher loop contributions are significant at such temperatures.

4. OPTIMIZED EXPANSION

The optimized expansion [12] consists in introducing an arbitrary parameter Ω into the Lagrangian density:

$$L_\mu^\epsilon[\Phi, \Omega] = \Phi^*(\mathbf{r}, \tau)\left(-\frac{\partial}{\partial\tau} + \frac{\hbar^2}{2m}\nabla^2 + \Omega\right)\Phi(\mathbf{r}, \tau) + \epsilon[\Phi^*(\mathbf{r}, \tau)(\mu - \Omega)\Phi(\mathbf{r}, \tau) - \lambda(\Phi^*(\mathbf{r}, \tau)\Phi(\mathbf{r}, \tau))^2]. \quad (26)$$

For $\epsilon = 1$, the dependence on Ω cancels and the modified Lagrangian coincides with the original one (4).

Calculating $Z[J]$ and performing the Legendre transform yields the effective action as a series in a formal parameter ϵ. The exact result does not depend on Ω; however, such a dependence appears in the nth order truncation, $\Gamma^{(n)}[\Phi, \Omega]$, obtained after setting $\epsilon = 1$. We exploit this freedom by choosing the value of Ω that fulfils the minimal sensitivity requirement

$$\left.\frac{\delta\Gamma^{(n)}}{\delta\Omega}\right|_{\Omega^{\text{opt}}} = 0 \tag{27}$$

in the given order approximation $\Gamma^{(n)}[\Phi, \Omega^{\text{opt}}]$. The optimal value of Ω changes from order to order, improving the convergence properties of the scheme. The approach is equivalent to a systematic resummation of the perturbation series.

The Gaussian approximation to the effective action is obtained in the first order of the optimized expansion. This yields the Gaussian effective potential

$$\begin{aligned} V^{(1)}[\Phi, \Omega] &= -\mu|\Phi|^2 + \lambda|\Phi|^4 + I_1(\Omega, |\Phi|^2) \\ &+ (\Omega - \mu)I_0(\Omega, |\Phi|^2) + 3\lambda I_0^2(\Omega, |\Phi|^2), \end{aligned} \tag{28}$$

with the self-consistency condition

$$\frac{dV^{(1)}}{d\Omega} = [\Omega - \mu + 6\lambda I_0(\Omega, |\Phi|^2)]\frac{dI_0}{d\Omega} = 0, \tag{29}$$

where I_1 and I_0 are defined by (19) and (23), respectively. Since both the effective potential and the optimization condition depend only on $|\Phi|^2$, one clearly sees that the H–P theorem is respected. The symmetry is spontaneously broken at $|\Phi|^2 = n_0$, which renders the effective potential stationary:

$$\begin{aligned} \left.\frac{dV^{(1)}}{d|\Phi|^2}\right|_{|\Phi|^2 = n_0} &= -\mu + 2\lambda n_0^2 + \lambda J_0(\Omega, n_0) \\ + [\Omega - \mu + 6\lambda I_0(\Omega, n_0)]&\left.\frac{dI_0(\Omega, |\Phi|^2)}{d|\Phi|^2}\right|_{|\Phi|^2 = n_0} = 0, \end{aligned} \tag{30}$$

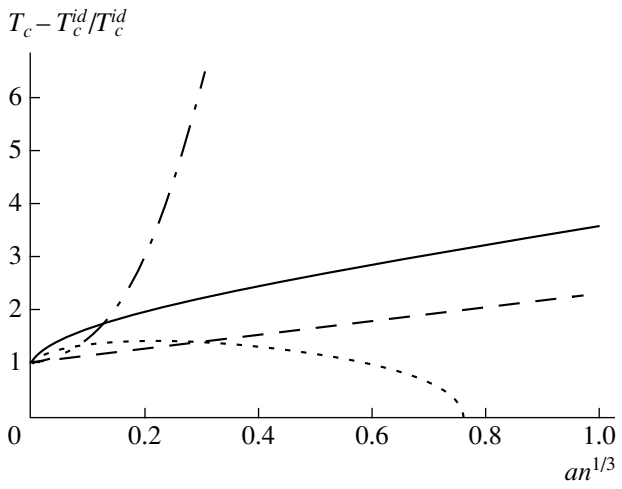
with J_0 given by (21). The particle density in our approximation reads

$$n = -\frac{dV^{(1)}}{d\mu} = n_0 + I_0(\Omega, n_0). \tag{31}$$

Upon eliminating μ using Eq. (29), Eqs. (30) and (31) simplify to

$$\begin{aligned} n &= n_0 + I_0(\Omega, n_0), \\ \Omega &= 2\lambda n_0^2 + \lambda J_0(\Omega, n_0) - 6\lambda I_0(\Omega, n_0), \end{aligned} \tag{32}$$

which, after numerically eliminating Ω, determine $n_0(n, \beta)$.



The critical temperature for BEC in the Gaussian approximation (solid line) compared with the 1-loop result [16] (dotted line) and the results of the 3-dimensional theory: the leading order in $an^{1/3}$ [15] (dashed line) and next-to-leading order [14] (dotted-dashed line).

5. RESULTS AND CONCLUSIONS

The critical temperature for BEC, T_c, is the temperature below which the symmetry is spontaneously broken. Its value can be calculated from the condition that $|\Phi|^2 = n_0 = 0$ at the phase transition. For the ideal gas $T_c^{id} = \left(\frac{2\pi}{m}\right)\left(\frac{n}{\zeta\left(\frac{3}{2}\right)}\right)^{\frac{2}{3}}$, calculating the shift of the critical temperature in the presence of interactions

$$\Delta T_c = \frac{T_c - T_c^{id}}{T_c^{id}} \tag{33}$$

generates controversy, even for a dilute gas. Different powers of the leading behavior in $an^{1/3}$ with different coefficients have been reported by various authors [14]. T_c is usually derived from the effective three-dimensional theory, arguing that only the zero Matsubara modes determine the critical behavior. In this approximation, the leading behavior $\Delta T_c \approx 1.32an^{1/3}$ [15] and next-to-leading order corrections have been determined [14].

Here, we show the approximate results for T_c derived in the original theory in (3 + 1)-dimensions. The 1-loop approximation for T_c has been calculated [16] by setting the value of the background field n_0 to zero in Eq. (24), which yields the equation

$$n = I_0(2\lambda n, n), \tag{34}$$

to be solved numerically. We calculate the Gaussian approximation for T_c by setting $n_0 = 0$ in Eq. (35), which results in the coupled pair of equations

$$\begin{aligned} n &= I_0(\Omega, 0), \\ \Omega &= \lambda J_0(\Omega, 0) - 6\lambda I_0(\Omega, 0). \end{aligned} \tag{35}$$

In the figure, the numerical results for T_c are compared with the results derived in the dimensionally reduced theory [14, 15]. For the experimental evaluation of T_c, the accuracy achieved in homogeneous systems is still insufficient. The theoretical results differ greatly, even at small values of $an^{1/3}$. It is interesting to observe that the leading behavior in the 1-loop approximation is of the form $\Delta T_c \approx \frac{4\sqrt{\pi}}{3}\frac{\sqrt{an^{1/3}}}{[\zeta(3/2)]^{2/3}}$ (with the sign opposite to that obtained by Toyoda [9]), while the Gaussian approximation gives $\Delta T_c \approx \frac{4\sqrt{\pi}}{3}\frac{\sqrt{an^{1/3}}}{[\zeta(3/2)]^{2/3}}$ with the coefficient $\sqrt{2}$ times larger than 1-loop result. In both cases, a square root dependence on $an^{1/3}$ is obtained that is similar to result obtained by Lee and Yang [2] but differs in the linear dependence derived from the dimensionally reduced theory. The behavior of T_c in the Gaussian approximation at larger values of $an^{1/3}$ becomes qualitatively different from the 1-loop behavior and there is no sign of the re-entrant phase transition suggested [16] by the 1-loop approximation. It would be interesting to investigate this issue in higher order approximations. Both the loop expansion and optimized expansion offer a systematic and consistent way to perform such a study without violating the H–P theorem.

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