Nonstandard expansion techniques for the finite-temperature effective potential in $\lambda\phi^4$ quantum field theory

Anna Okopińska
Institute of Physics, Warsaw University, Białystok Branch, 15-424 Białystok, Lipowa 41, Poland
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The finite-temperature effective potential in scalar quantum field theory in n-dimensional space-time is calculated using different expansions of a path-integral representation for the partition function. First-order results in an optimized and mean-field expansion are compared with a conventional one-loop result. In one dimension the comparison with the finite-temperature effective potential calculated numerically shows that only the mean-field method gives qualitative agreement in the entire range of Lagrangian parameters. In four dimensions the mean-field method seems also to be most reliable. The thermal properties of the renormalized theory indicate that a scalar theory is noninteracting.

I. INTRODUCTION

The interest in thermodynamic properties of quantum field systems started after Kirzhnits and Linde[1] suggested that the spontaneous symmetry breaking induced by scalar fields will disappear beyond a critical temperature $T_{\rm cr}$ and the Universe in the big bang cosmology can undergo a series of phase transitions. They can be studied with the help of finite-temperature generating functionals in the "imaginary time" formalism.[2] The generating functional for Green's functions in quantum field theory (QFT) can be defined as the path integral

$$Z[J]=\int D\Phi \exp\left[-S[\Phi]+\int d_n x\, J(x)\Phi(x)\right], \quad (1.1)$$

where $S[\Phi]$ denotes a classical action and $d_n x$ is a volume element in n-dimensional Euclidean space. For definiteness we fix a normalization of the functional measure to have

$$\int D\Phi \exp\left[-\int d_n x\, d_n y\, \Phi(x) A(x,y)\Phi(y)\right] = \mathrm{Det}^{-1/2} A \quad (1.2)$$

for a positive-definite operator A. The effective action is defined as

$$\Gamma[\phi]=W[J]-\int d_n x\, J(x)\phi(x), \quad (1.3)$$

where $W[J]=\ln Z[J]$ is the connected generating functional, and the background field

$$\phi(x)=\frac{\delta W}{\delta J(x)} = \int D\Phi\, \Phi(x) \exp\left[-S[\Phi]+\int d_n x\, J(x)\Phi(x)\right] \quad (1.4)$$

is a vacuum expectation value (VEV) of the scalar field for given $J(x)$. The effective potential is defined for space-time constant fields as

$$V(\phi)=-\frac{\Gamma[\phi]\,|_{\phi(x)=\phi}}{\int d_n x}, \quad (1.5)$$

and is stationary for vanishing J, due to Legendre transform properties. The above functionals can be defined for QFT or for equilibrium thermodynamics, if the appropriate boundary conditions in the path integral (1.1) are chosen. The generating functionals for Green's functions in QFT are obtained, if the integral is taken over the functions in n-dimensional Euclidean space, which vanish at $x_0=\pm\infty$. In this case, Eq. (1.5) defines the effective potential (EP). The partition function at the temperature T, generalized to include an interaction with external source $J(x)$, is given by the same integral (1.1), but taken over the functions in n-dimensional Euclidean space identified with a period T^{-1} in "imaginary-time coordinate" x_0. In this case Eq. (1.5) defines the finite-temperature effective potential (FTEP). It has a meaning of the free energy density, i.e., the work required by the external source J to displace the VEV of the scalar field from $\phi(0)$ to $\phi(J)$ in the unit space volume at given temperature. For $T=0$ the FTEP becomes equal to the ordinary EP.

We limit ourselves to a real scalar field with the classical action given by

$$S[\Phi]=\int d_n x\left[\tfrac{1}{2}\Phi(x)(-\partial^2+m^2)\Phi(x)+\lambda\Phi^4(x)\right]. \quad (1.6)$$

Usually the temperature-dependent effective potential is calculated in a loop expansion (LE) (Ref. 2). If $m^2<0$, the one-loop FTEP is double-well shaped for $T<T_{\rm cr}$, and for $T>T_{\rm cr}$ it has only one minimum at $\phi=0$. This is used as an argument that the reflection symmetry in four dimensions is broken spontaneously only below $T_{\rm cr}$. However, the LE breaks down for the values of ϕ in the region where the resulting FTEP is nonconvex.[3] In this case, the convex hull of the FTEP (Maxwell construction) should be taken to minimize the energy density of the whole system, as explained in Ref. 4 for $T=0$. A phase transition can be an artifact of the approximation, and the LE can be misleading in the investigation of cosmological models of the early Universe.

In this work we will study the applicability of various expansions, generated if the steepest-descent method is

applied to the path integral (1.1) with the subintegral exponent represented in various forms. Such a method has been used in my previous work[5] to discuss the (a) loop expansion (LE), (b) optimized expansion (OE), (c) mean-field (MF) expansion, for the EP in scalar QFT in arbitrary-dimensional Euclidean space. In this paper we will study the above expansions at finite temperature. The FTEP is obtained in the same way as the EP, the only difference is due to periodic boundary conditions, as will be discussed in Sec. II.

In Sec. III we present the results in one-dimensional space-time when QFT becomes the quantum mechanics of an anharmonic oscillator. The FTEP calculated to first order in the above approximations is compared with the exact result calculated numerically. For $m^2>0$ the results are very similar and agree well with the exact FTEP; however, for $m^2\leq 0$ only the MF method gives a qualitative agreement with the exact result.

In Sec. IV the renormalized FTEP in four-dimensional space-time in the leading-order MF is compared with the first-order OE result. Two possibilities emerging for the renormalized $\lambda\phi^4$ QFT, triviality or precariousness, are discussed.

II. FORMAL METHODS OF EVALUATION OF THE FINITE-TEMPERATURE EFFECTIVE POTENTIAL

The generating functional $Z[J]$ given as the path integral (1.1) can be calculated in the steepest-descent method. It consists in introducing a formal parameter in such a way that an expansion of the subintegral expression produces Gaussian integrals. After evaluating them, $Z[J]$ is obtained as a series in a formal parameter. The loop expansion, optimized expansion, and mean-field expansion in scalar QFT have been discussed[5] as examples of such a procedure. For constant background fields Fourier transforms can be done and the effective potential can be expressed in terms of ordinary integrals in momentum space. For zero temperature the integral (1.1) is taken over all functions in Euclidean space; therefore, the Fourier transform is expressed as an integral over

$$\int \frac{d_n p}{(2\pi)^n} \ . \tag{2.1}$$

In equilibrium thermodynamics the integral in (1.1) is taken over periodic functions with period T^{-1}; therefore, the momentum vector p has an "imaginary time" component $p_0=2\pi jT$, where $j=0,\pm 1,\ldots,$ and a Fourier transform becomes a combined sum over j and integral in $\nu=n-1$ dimensional momentum space

$$T\sum_j \int \frac{d_\nu p}{(2\pi)^\nu} \ . \tag{2.2}$$

The structure of the FTEP in an approximation generated by the steepest-descent method is formally identical, as for the ordinary EP. Therefore, the FTEP can be obtained after replacing the integrals (2.1) by (2.2) in the given order approximation for the EP. The integrals appearing frequently at zero temperature,

$$\begin{aligned} I_1(\Omega)&=\frac{1}{2}\int \frac{d_n p}{(2\pi)^n}\ln(p^2+\Omega^2)=\int \frac{d_\nu p}{(2\pi)^\nu}\frac{\omega_p}{2} \ , \\ I_0(\Omega)&=\int \frac{d_n p}{(2\pi)^n}\frac{1}{p^2+\Omega^2}=\int \frac{d_\nu p}{(2\pi)^\nu}\frac{1}{2\omega_p} \ , \end{aligned} \tag{2.3}$$

where $\nu=n-1$ and $\omega_p=\Omega^2+p_1{}^2+p_2{}^2+\cdots+p_\nu{}^2$, will be replaced by their finite-temperature counterparts

$$\begin{aligned} I_1^T(\Omega)&=\tfrac{1}{2}T\sum_j \int \frac{d_\nu p}{(2\pi)^\nu}\ln(p^2+\Omega^2) \\ &=\int \frac{d_\nu p}{(2\pi)^\nu}\left[\frac{\omega_p}{2}+T\ln(1-e^{-\omega_p/T})\right] \\ &=I_1(\Omega)+J_1^T(\Omega) \ , \\ I_0^T(\Omega)&=T\sum_j \int \frac{d_\nu p}{(2\pi)^\nu}\frac{1}{p^2+\Omega^2} \\ &=\int \frac{d_\nu p}{(2\pi)^\nu}\frac{1}{\omega_p}\left[\frac{1}{2}+\frac{1}{e^{\omega_p/T}-1}\right] \\ &=I_0(\Omega)+J_0^T(\Omega) \ . \end{aligned} \tag{2.4}$$

The temperature-dependent components $J_1^T(\Omega)$ and $J_0^T(\Omega)$ are finite for any space dimension ν and vanish for $T=0$.

A. Loop expansion

The loop expansion is generated if the steepest-descent method is applied to the generating functional written in the form

$$Z[J]=\int D\Phi \exp\left[\frac{1}{\hbar}(-S[\Phi]+J\Phi)\right] \ , \tag{2.5}$$

with the Planck constant, retained in the exponent, as a formal parameter of expansion.[2] The temperature-dependent effective potential to the order $\hbar$ can be expressed as

$$V_1^T(\phi)=\frac{m^2}{2}\phi^2+\lambda\phi^4+\hbar I_1^T(\Omega) \ , \tag{2.6}$$

where $\Omega^2=m^2+12\lambda\phi^2$ and I_1^T is defined in (2.4). For $T=0$ the FTEP becomes an ordinary one-loop EP. For $m^2<0$ the FTEP at $\phi^2<-m^2/12\lambda$ is complex for all temperatures, which is a signal of the breaking of the LE.

B. Optimized expansion

The optimized expansion[5] (OE) is generated applying the steepest-descent method to the integral (1.1) with the classical action written as

$$\begin{aligned} S_\epsilon[\Phi]&=S^{(0)}[\Phi]+\epsilon S^{(1)}[\Phi] \\ &=\tfrac{1}{2}\Phi(-\partial^2+\Omega^2)\Phi \\ &\quad+\epsilon[\tfrac{1}{2}(m^2-\Omega^2)\Phi^2+\lambda\Phi^4] \ . \end{aligned} \tag{2.7}$$

The parameter ϵ is a formal parameter of expansion and is set equal to one at the end; hence $Z[J]$ does not depend on an arbitrary field $\Omega(x)$. However, if the series is truncated to a finite order in ϵ, the dependence on Ω appears. Guided by Stevenson's principle of minimal sensitivity[6] we require the kth-order approximant of the FTEP to be as insensitive as possible to the small variation of the unphysical parameter, choosing Ω to satisfy

$$\frac{\partial V_k^T}{\partial \Omega}=0 . \tag{2.8}$$

The optimized FTEP can be obtained replacing the momentum integrals in the optimized EP (Ref. 5) by their finite-temperature counterparts. To first order we obtain

$$V_1^T(\phi,\Omega)=\frac{m^2\phi^2}{2}+\lambda\phi^4+I_1^T(\Omega) +\tfrac{1}{2}(m^2+12\lambda\phi^2-\Omega^2)I_0^T(\Omega) +3\lambda[I_0^T(\Omega)]^2 , \tag{2.9}$$

with Ω satisfying the "gap equation" obtained from (2.8):

$$\Omega^2-m^2-12\lambda[\phi^2+I_0^T(\Omega)]=0 . \tag{2.10}$$

The consistency of the method requires $\Omega^2>0$, leading to a real FTEP. For $T=0$ the FTEP given by Eqs. (2.9) and (2.10) becomes the ordinary Gaussian effective potential (GEP) obtained from the Ritz variational principle for Gaussian trial functions.[7,8] For $T\neq 0$ our first-order result coincides with the temperature-dependent GEP obtained by Roditi[9] from the Bogoliubov variational principle, requiring the expectation value of the free energy in the Gaussian trial state to be minimal. The same result can also be obtained in the "real-time" formalism.[10] It is a generalization of the finite-temperature Hartree approximation in quantum mechanics[11] (QM) for QFT in arbitrary dimensional space-time.

The OE can be regarded as a systematic method to improve variational results at finite temperature. There is, however, a difference between first-order OE and variational result. The variational parameter Ω is fixed to minimize the expectation value of some operator; therefore, the result should be compared with its values at end points. The first-order OE gives the same function of Ω but the principle of minimal sensitivity requires only the dependence on the unphysical parameter to be flat. Any of the stationary points can be chosen equally well, and the convergence of the method can be checked only *a posteriori*. The variational principle argument, that the value of Ω corresponding to a minimum should be chosen, cannot by applied beyond first-order OE.

C. Mean-field theory

The mean-field (MF) expansion, proposed by Cooper, Guralnik, and Kasdan,[12] can be formulated in a path-integral approach.[13] The generating functional after introducing an auxiliary field $\theta(x)$ and its source $S(x)$ can be expressed as

$$Z[J,S]=\int D\theta \exp\left[-\frac{1}{16\lambda}(\theta^2-2im^2\theta)+\frac{m^4}{16\lambda}+S\theta\right]\mathrm{Det}^{1/2}(8\lambda I)\int D\Phi \exp[-\tfrac{1}{2}\Phi(-\partial^2-i\theta)\Phi+J\Phi] , \tag{2.11}$$

where the space arguments and the integrations over them are suppressed. Upon performing the integration over Φ we have

$$Z[J,S]=\mathrm{Det}^{1/2}(8\lambda I)\int D\theta \exp(-NF[\theta,J,S]) , \tag{2.12}$$

where

$$F[\theta,J,S]=-\frac{m^4}{16\lambda}+\frac{1}{16\lambda}\theta^2-\frac{1}{8\lambda}im^2\theta -\tfrac{1}{2}JGJ+\tfrac{1}{2}\mathrm{Tr}LnG^{-1}-S\theta \tag{2.13}$$

and

$$G^{-1}(x,y)=[-\partial^2-i\theta(x)]\partial(x-y) . \tag{2.14}$$

$Z[J,S]$ can be calculated as a series in a formal parameter $1/N$, which is introduced to identify the orders of the expansion and is set equal to one at the end. For an N-component scalar field a number of components appear in (2.12) naturally, leading to large-N expansion. The MF result for one real scalar field can be formally obtained by setting $N=1$ in the given order of the large-N expansion.

After replacing the sources by the background fields $\phi=\delta \ln Z/\delta J$ and $\Omega^2=-i\delta \ln Z/\delta S$ with the aid of a Legendre transform, the effective potential for constant background fields is obtained. The FTEP to leading (zeroth) order in N becomes

$$V_0^T(\phi,\Omega)=-\frac{m^4}{16\lambda}-\frac{1}{16\lambda}\Omega^4 +\frac{1}{8\lambda}\Omega^2(m^2+4\lambda\phi^2)+I_1^T(\Omega) \tag{2.15}$$

and to the first order in $1/N$ we have

$$V_1^T(\phi,\Omega)=-\frac{m^4}{16\lambda}-\frac{1}{16\lambda}\Omega^4+\frac{1}{8\lambda}\Omega^2(m^2+4\lambda\phi^2)+I_1^T(\Omega)$$
$$+\tfrac{1}{2}T\sum_J\int\frac{d_\nu p}{(2\pi)^\nu}\ln\left[1+\frac{8\lambda\phi^2}{p^2+\Omega^2}+4\lambda\sum_i\int\frac{d_\nu q}{(2\pi)^\nu}\frac{1}{(q^2+\Omega^2)[(q+p)^2+\Omega^2]}\right], \tag{2.16}$$

where $p_0=2\pi jT$, $q_0=2\pi iT$, and $i,j=0,\pm1,\ldots$. The VEV of the auxiliary field Ω in both cases has to be eliminated by the "gap equation":

$$\Omega^2-m^2-4\lambda\phi^2-4\lambda I_0^T(\Omega)=0\ , \tag{2.17}$$

obtained after setting the source $S=0$. The method is consistent if $\Omega^2>0$. The leading-order result (2.15) has been discussed as a large-N limit in Ref. 14. The next order (2.16) is obtained after replacing the integrals (2.1) in the expression for the EP (Ref. 5) by (2.2).

III. QUANTUM MECHANICS

The path-integral quantization enables the study of a field theory in arbitrary space-time dimension. The self-interacting scalar field theory in one dimension (time) is quantum mechanics of an anharmonic oscillator. In this case the exact FTEP can be calculated numerically, as a partition function of the anharmonic oscillator, placed in a constant electric field J, can be expressed as

$$Z(J)=\sum_n e^{-E_n(J)/T}\ , \tag{3.1}$$

where E_n are the eigenvalues of the Schrödinger equation

$$\left[-\frac{1}{2}\frac{d^2}{dx^2}+\frac{m^2}{2}x^2+\lambda x^4-Jx\right]\psi_n(x)=E_n(J)\psi_n(x)\ . \tag{3.2}$$

A linear variational method, using harmonic-oscillator wave functions with an appropriately chosen frequency,[15] has been used to calculate a set of eigenvalues necessary to sum (3.1) with an appropriate accuracy. Afterward, a Legendre transformation has been done numerically to obtain the FTEP.

When discussing the numerical results it is convenient to use the dimensionless quantities. After rescaling $V^T\to V^T\lambda^{1/3}$, $T\to T\lambda^{1/3}$, $\phi\to\phi\lambda^{-1/6}$ only one parameter $z=\frac{1}{2}m^2\lambda^{-2/3}$ remains. The FTEP to first order in the approximations studied in the previous section become as follows.

(a) A loop expansion:

$$V_1^T(\phi)=z\phi^2+\phi^4+\tfrac{1}{2}(2z+12\phi^2)^{1/2}$$
$$+T\ln(1-e^{-(2z+12\phi^2)^{1/2}/T})\ . \tag{3.3}$$

(b) An optimized expansion:

$$V_1^T(\phi)=z\phi^2+\phi^4+\frac{\Omega}{2}+T\ln(1-e^{-\Omega/T})$$
$$-\frac{\Omega^2-2z-12\phi^2}{2\Omega}\left[\frac{1}{2}+\frac{1}{e^{\Omega/T}-1}\right]$$
$$+\frac{3}{\Omega^2}\left[\frac{1}{2}+\frac{1}{e^{\Omega/T}-1}\right]^2, \tag{3.4}$$

with Ω satisfying the equation

$$\Omega^3=2(z+6\phi^2)\Omega-12\left[\frac{1}{2}+\frac{1}{e^{\Omega/T}-1}\right]=0\ . \tag{3.5}$$

(c) A mean-field expansion:

$$V_1^T(\phi)=-\frac{z^2}{4}-\frac{\Omega^4}{16}+\frac{\Omega^2}{4}(z+2\phi^2)-\Omega-T\ln(1-e^{-2\Omega/T})$$
$$+\tfrac{1}{2}\Omega_1+T\ln(1-e^{-\Omega_1/T})$$
$$+\tfrac{1}{2}\Omega_2+T\ln(1-e^{-\Omega_2/T})\ , \tag{3.6}$$

where

$$\Omega_{1,2}=\frac{1}{\sqrt{2}}\left\{5\Omega^2+8\phi^2+\frac{4}{\Omega}\right.$$
$$\left.\pm\left[\left(5\Omega^2+8\phi^2+\frac{4}{\Omega}\right)^2-16\Omega^4+8\Omega^2\phi^2+\Omega\right)\right]^{1/2}\Bigg\}^{1/2}\ ,$$

and Ω is a positive root of the equation

$$\Omega^3-2(z+2\phi^2)\Omega-4\left[\frac{1}{2}+\frac{1}{e^{\Omega/T}-1}\right]=0\ . \tag{3.7}$$

For $z>0$ the results for the FTEP obtained in the above approximations are very similar and approach the exact result for $z\to\infty$. For decreasing z the differences between approximations increase and become very marked for negative z. In Figs. 1, 2, and 3 the exact results at $z=-10$ are compared with the FTEP in the first order of the LE, OE, and MF for a range of temperatures. Only one-half of the symmetric plot is shown. As there is no spontaneous symmetry breaking in QM at any temperature, the exact FTEP has only one minimum

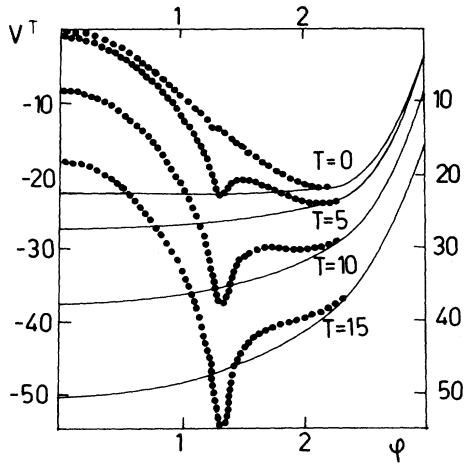

FIG. 1. The one-loop FTEP (dotted line) for quantum-mechanical anharmonic oscillator with $z=-10$ compared with the exact result (solid line) at temperature $T=0,5,10,15$.

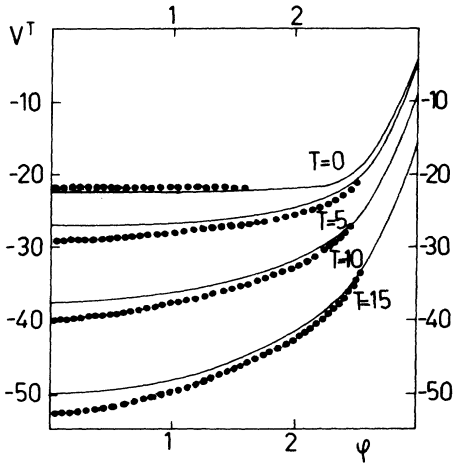

FIG. 3. As Fig. 1, but for first order of the mean-field expansion (dotted line).

at $\phi=0$ and the exact free energy for vanishing source is equal to the value of the FTEP at this point.

The one-loop FTEP is complex for $\phi^2<\phi_1{}^2=-z/6$ and the approximation is expected to be poor in this region. The real part of the one-loop FTEP plotted in Fig. 1 has minima at $\pm\phi_1$ and $\pm\phi_2$, its value at $\phi=\phi_2$ gives a much better approximation of the free energy than the value at $\phi=0$. The Maxwell construction between minima at $\pm\phi_2$, which excises minima at $\pm\phi_1$, makes an agreement with the exact FTEP quite good. However, beyond a critical temperature ($T_{\rm cr}^{\rm LE}\simeq 10.96$ for $z=-10$), the minima at $\pm\phi_2$ disappear and the Maxwell construction cannot be done. Therefore, the LE can be used only below $T_{\rm cr}^{\rm LE}$ and one-loop FTEP should be improved by Maxwell construction. For $z<z_{\rm cr}=-3.25$ a critical temperature becomes negative and the LE is not applicable even at $T=0$ (see Ref. 5).

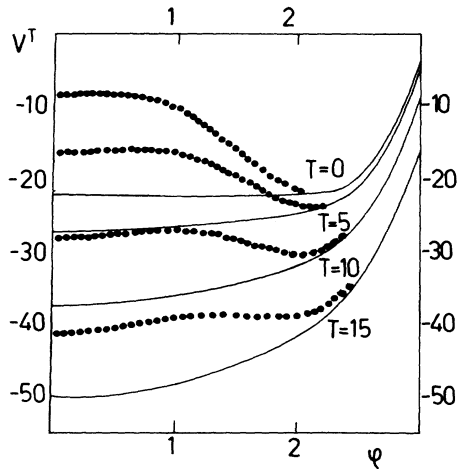

FIG. 2. As Fig. 1, but for first order of the optimized expansion (dotted line).

The FTEP to first order in the OE coincides with the temperature-dependent Hartree free energy, which has been shown[11] to be in good agreement with the exact result for $z>0$. For $z<0$ the OE becomes much worse, as can be seen in Fig. 2 for $z=-10$. At low temperatures, apart from minimum at $\phi=0$ a minimum at ϕ_2 appears as discussed in Ref. 10. Only after the Maxwell construction does the FTEP agree with the exact one. Beyond a critical temperature ($T_{\rm cr}^{\rm OE}\simeq 16.57$ for $z=-10$) the FTEP becomes convex in agreement with the exact result. Therefore, the OE can be applied for the entire range of temperatures; below $T_{\rm cr}^{\rm OE}$ the result should be improved by the Maxwell construction. For $z>z_{\rm cr}=-2.45$ a critical temperature becomes negative, the FTEP is convex even at $T=0$ (see Refs. 16 and 5), and the OE gives a qualitatively good description of the free energy.

The first-order MF result, shown in Fig. 3, is convex and agrees with the exact FTEP in the whole range of temperatures. The agreement becomes lessened with increasing temperature; however, qualitative description of the FTEP remains reasonable.

The LE can be used only below $T_{\rm cr}^{\rm LE}$, and should be improved by the Maxwell construction. The OE can be used in the whole range of temperatures; however, below $T_{\rm cr}^{\rm OE}$ it also requires the Maxwell construction. It should be stressed that the Maxwell construction provides a good approximation of the numerical FTEP, but it suggests a phase transition, which does not take place in QM. Therefore, the only method which gives qualitatively good description of the features of the FTEP in QM is the MF method.

IV. QUANTUM FIELD THEORY

If the dimension of the space-time is greater than one the integrals in momentum space become divergent and should be regularized, for example, with the UV cutoff Λ. To obtain a finite and Λ-independent content, a

theory should be reparametrized in terms of physical quantities, instead of bare parameters. The renormalized mass and coupling constants for scalar QFT can be defined as the second and fourth derivative of the EP at its stationary point. For $\lambda\phi^4$ theory there are arguments that the renormalized theory is noninteracting.[17] However, renormalization in the LE can be accomplished expressing the FTEP in terms of scalar particle mass and self-interaction constant,[2] without any signal about triviality. In the OE and MF the situation is different.

Renormalization of the GEP, which is equivalent to the first-order OE result at zero temperature, has been studied by Stevenson[8] with the renormalized mass and coupling constant defined as

$$m_R{}^2=\frac{d^2V_1}{d\phi^2}\bigg|_0=m^2+12\lambda I_0(m_R)_0\ , \tag{4.1}$$

$$\lambda_R=\frac{1}{24}\frac{d^4V_1}{d\phi^4}\bigg|_0=\lambda\frac{1-12\lambda I_{-1}(m_R)}{1+6\lambda I_{-1}(m_R)}\ , \tag{4.2}$$

where

$$I_{-1}(\Omega)=2\int\frac{d_np}{(2\pi)^n}\frac{1}{(p^2+\Omega^2)^2}=\int\frac{d_vp}{(2\pi)^v}\frac{1}{2\omega_p{}^3} \tag{4.3}$$

diverges in the limit of $\Lambda\to\infty$. The renormalization condition (4.2) in the limit of infinite cutoff has two solutions for λ:

$$\lambda=-\tfrac{1}{2}\lambda_R \tag{4.4}$$

or

$$\lambda=-\frac{1}{6I_{-1}(m_R)}\left[1+\frac{1}{2\lambda_R I_{-1}(m_R)}\right]\ . \tag{4.5}$$

The bare mass can be eliminated with the aid of (4.1) and the bare coupling λ by (4.4) or (4.5).

If λ is eliminated by (4.4), the first-order FTEP (2.9) after using the "gap equation" (2.10) becomes

$$V_1^T(\phi)=\tfrac{1}{2}m_R{}^2\phi^2-2\lambda\phi^4+J_1^T(m_R)\ . \tag{4.6}$$

For $T=0$ the result approaches the EP obtained by Stevenson.[8] He has argued that the theory is unstable, as for $\lambda>0$ the EP (4.6) is unbounded below for large ϕ, and for $\lambda<0$ the global minimum is given by the EP at the $\Omega=0$ end point, which is also unbounded below. However, we have shown[5] that the instability may not be a feature of the theory, but can be introduced in the reparametrization procedure; as for $\Omega=0$ the OE breaks down and the derivatives (4.1) and (4.2) are determined poorly. The finite-temperature behavior of (4.6) indicates that a theory is noninteracting, as the temperature contribution is a ϕ-independent term, which is the same as for a free particle with mass m_R.

If the infinitesimal negative solution (4.5) λ is used to eliminate a bare coupling, the first-order FTEP becomes[9]

$$V_1^T(\phi)=\tfrac{1}{2}\Omega^2\phi^2-\frac{1}{16\lambda_R}(\Omega^2-m_R{}^2)^2+\frac{1}{32\pi^2}m_R{}^4L_3(x)+J_1^T(\Omega)\ , \tag{4.7}$$

where Ω satisfies

$$\tfrac{1}{2}\phi^2-\frac{1}{8\lambda_R}(\Omega^2-m_R{}^2)+\frac{1}{32\pi^2}m_R{}^2L_2(x)+\tfrac{1}{2}J_0^T(\Omega)\ , \tag{4.8}$$

with $L_2(x)=x\ln x-x+1$, $L_3(x)=\frac{1}{4}[2x^2\ln x-2(x-1)-3(x-1)^2]$, and $x=\Omega^2/m_R{}^2$. For $T=0$, the temperature-dependent contributions vanish and the result becomes a precarious EP obtained by Stevenson.[8] In Fig. 4 we compare the FTEP at $T=0.3$ with his result, using the same units of m_R and taking $\kappa=-4\pi^2/\lambda_R=1.25$, as in Fig. 7 of Ref. 8. At $T=0$ there is a critical value $\phi_{cr}=0.058$, beyond which there is no solution to the "gap equation" (4.8) and the end-point value $\Omega=0$, which gives constant EP should be taken instead. For increasing temperature ϕ_{cr} decreases and becomes equal to zero for $T=0.41$. The critical temperature is even lower, as for $T_{cr}=0.35$ the value of the FTEP at the end point becomes lower than the value obtained using the gap equation. Beyond the critical temperature the FTEP is a constant potential of massless free theory, as foreseen by Stevenson from strong-coupling behavior of a precarious theory. However, if $\Omega=0$ a reparametrization in the OE introduces big errors into FTEP. Therefore, the behavior beyond T_{cr} must not be necessarily a feature of the precarious theory, but of the wrong approximation.

In the MF expansion renormalization has to be done already in the leading order. The renormalized vertices to this order become

$$m_R{}^2=\frac{d^2V_0}{d\phi^2}\bigg|_0=m^2+4\lambda I_0(m_R)\ , \tag{4.9}$$

$$\lambda_R=\frac{1}{24}\frac{d^4V_0}{d\phi^4}\bigg|_0=\frac{\lambda}{1+2\lambda I_{-1}(m_R)}\ . \tag{4.10}$$

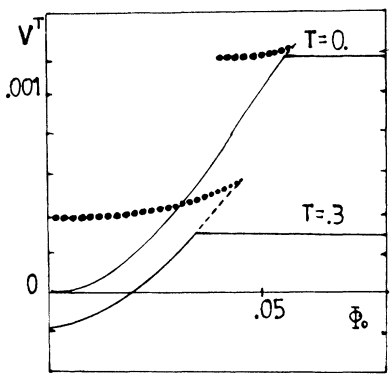

FIG. 4. The precarious FTEP (solid line) for scalar QFT in four dimensions with $\kappa=-4\pi^2/\lambda_R=1.25$, at $T=0$ and $T=0.3$. The dotted and dashed lines show the FTEP corresponding to local minima of the renormalized gap equation.

In this approximation there are also two possibilities. Keeping the bare coupling constant λ fixed, the renormalized coupling constant approaches zero and we obtain

$$V_0^T(\phi)=\tfrac{1}{2}m_R{}^2\phi^2+J_1^T(m_R)\ , \tag{4.11}$$

which is the FTEP of a free particle with a mass m_R. If the bare coupling constant is taken to be infinitesimal and negative,

$$\lambda=-\frac{1}{2I_{-1}(m_R)}\left[1+\frac{1}{2\lambda_R I_{-1}(m_R)}\right]\ , \tag{4.12}$$

we obtain the same precarious FTEP as in the first-order OE (4.7) and (4.8). The results of the MF method are similar, as obtained by Bardeen and Moshe in variational treatment of the large-N expansion.[14]

The results of the leading-order MF and of the first-order OE show that the QFT of the neutral scalar field is trivial in agreement with the rigorous result.[17] In the MF method a triviality of the theory can be seen from the behavior of the EP. In the OE consideration of thermal properties is necessary to draw this conclusion. Therefore, the MF seems to be the most reliable of the considered method, just as it is in lower-dimensional space-time.

Both methods indicate that an interacting theory can be obtained only if a bare coupling is chosen to be infinitesimal negative. However, a finite-temperature behavior of precarious theory suggests a phase transition to a free energy of massless particle.[8] This strange behavior above a critical temperature was attributed to internal instability due to negative bare coupling.[14] However, it should be stressed that the conclusion is drawn using the end-point behavior of the FTEP when the OE, as well as the MF, are unreliable. The lack of a solution to the gap equation is only a signal that both methods break down beyond a critical temperature.

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