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# Multi-particle States from the Effective Action for Local Composite Operators: Anharmonic Oscillator

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The effective action for the local composite operator $\Phi^2(x)$ in the scalar quantum field theory with $\lambda\Phi^4$ interaction is obtained in the expansion in two-particle-point-irreducible (2PPI) diagrams up to five-loops. The effective potential and 2-point Green's functions for elementary and composite fields are derived. The ground state energy as well as one- and two-particle excitations are calculated for space-time dimension $n=1$, when the theory is equivalent to the quantum mechanics of an anharmonic oscillator. The agreement with the exact spectrum of the oscillator is much better than that obtained within the perturbation theory. 

## 1. INTRODUCTION

The formalism of the effective action (EA) [1] is usually used in relativistic quantum field theory; hovewer the formulation is universal and provides an effective approach to any quantum theory. Here we consider the theory of a real scalar field in $n$-dimensional Euclidean space-time with a classical action given by

$$S[\Phi] = \int [\tfrac{1}{2}\Phi(x)(-\partial^2 + m^2)\,\Phi(x) + \lambda\Phi^4(x)]\,d^n x. \tag{1}$$

The simplest case of $n=1$ dimensional space-time, which is equivalent to the quantum mechanics of the anharmonic oscillator (AO), is frequently used as a testing ground for various field-theoretical methods. Here we shall discuss the method of the EA for local composite operators which provides a systematic approximation scheme for vacuum energy and lowest multi-particle excitations. We shall keep the dimension of the space-time $n$ arbitrary as long as possible, setting $n=1$ only in the last stage, where the energies are calculated.

The conventional EA, which is a generating functional for one-particle-irreducible (1-PI) Green's functions, is obtained by introducing a source coupled to the quantum field $\Phi(x)$. By coupling external sources to bilocal $\Phi(x)\,\Phi(y)$ [2] and local $\Phi^2(x)$ [3] fields, the generating functionals for composite operators are defined (for review see Ref. [4]). For an interacting theory the exact form of any functional is not known, so one resorts to approximations. Formulating an



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approximation scheme for a generating functional, a consistent set of approximate Green's functions can be obtained through differentiation. This is crucial for a relativistic quantum field theory, where the process of renormalization has to be performed. Any functional, if calculated exactly, contains the same information and gives the same result for a physical quantity; however, the same approximation scheme for various functionals would result in different approximations of Green's functions and observables. Therefore, an appropriate choice of a generating functional for calculating a quantity of interest is important. The conventional EA is used for discussing a vacuum structure and one-particle excitations. For a simultaneous study of two-particle excitations, the EA for composite operators is more suitable, since it determines the conventional EA (by eliminating the expectation values of composite operators) and generates Green's functions related directly to one- and two-particle eigenmodes.

Generating functionals can be calculated in the loop expansion. The conventional EA, $\Gamma[\varphi]$, is given by a sum of one-particle irreducibile vacuum diagrams [1]. The EA for the bilocal composite operator, $\Gamma[\varphi(x), G(x, y)]$ is given by two-particle-irreducibile (2PI) diagrams [2]. The one-loop result, after eliminating the full propagator $G(x, y)$, gives the Gaussian approximation for the conventional EA; however, beyond one-loop the gap equation for $G(x, y)$ is a highly non-trivial integral equation. The EA for the local composite operator, $\Gamma[\varphi(x), \Delta(x)]$, can be also obtained diagramaticaly [5, 6] and the one-loop result gives the Gaussian approximation. Calculations of post-Gaussian corrections are easier in this approach, since the gap equation for the vacuum expectation value of the local composite field is algebraic.

The vacuum functional for the composite operator $\Phi^2(x)$ is represented by a path integral

$$Z[J_1, J_2] = e^{W[J_1, J_2]} = \int D\Phi\, e^{-S[\Phi] + \int J_1(x)\, \Phi(x)\, d^n x + 1/2 \int J_2(x)\, \Phi^2(x)\, d^n x} \tag{2}$$

and the EA is obtained as a Legendre transform

$$\Gamma[\varphi, \Delta] = W[J_1, J_2] - \int J_1(x)\, \varphi(x)\, d^n x - \tfrac{1}{2} \int J_2(x)(\varphi^2(x) + \Delta(x))\, d^n x, \tag{3}$$

where

$$\frac{\delta W}{\delta J_1(x)} = \varphi(x), \qquad \text{and} \qquad \frac{\delta W}{\delta J_2(x)} = \frac{1}{2}\,(\varphi^2(x) + \Delta(x)) \tag{4}$$

determine the expectation values of the fields $\Phi$ and $\Phi^2$, in the presence of external currents $J_1$ and $J_2$. The EA fulfils

$$\frac{\delta \Gamma}{\delta \varphi(x)} = -J_1(x) - J_2(x)\, \varphi(x), \tag{5}$$

and

$$\frac{\delta \Gamma}{\delta \Delta(x)} = -\frac{1}{2} J_2(x). \tag{6}$$

Setting $J_1 = J_2 = 0$, which reproduces the physical theory (1), results in variational equations

$$\frac{\delta \Gamma}{\delta \varphi(x)} = 0 \tag{7}$$

and

$$\frac{\delta \Gamma}{\delta \Delta(x)} = 0. \tag{8}$$

These equations determine the vacuum expectation values $\varphi_0$ and $\Delta_0$, which are space-time independent, by translational invariance.

The conventional EA can be obtained as $\Gamma[\varphi] = \Gamma[\varphi, \Delta_0]$ with $\Delta_0[\varphi]$ determined by inverting the gap equation (8). The effective potential (EP), defined by

$$V(\phi) = -\frac{\Gamma[\varphi]|_{\varphi(x) = \phi = const}}{\int d^n x}, \tag{9}$$

gives the vacuum energy density $V(\varphi_0)$.

Green's functions, generated from the EA for local composite operators, provide a convenient tool to study multi-particle states, since their zero modes give directly the excitation energies above the ground state. One-particle eigenmode is determined by the 2-point Green's function for the elementary field

$$\Gamma^2(x-y) = \frac{\delta^2 \Gamma}{\delta \varphi(x)\, \delta \varphi(y)}\bigg|_{\varphi(x) = \varphi_0,\, \Delta(x) = \Delta_0}, \tag{10}$$

which is an inverse of the full propagator

$$W^2(x-y) = \langle T\Phi(x)\, \Phi(y) \rangle_{connected}. \tag{11}$$

An appropriate function to study two-particle excitation is the 2-point Green's function for the composite field

$$\Gamma^4(x-y) = \frac{\delta^2 \Gamma}{\delta \Delta(x)\, \delta \Delta(y)}\bigg|_{\varphi(x) = \varphi_0,\, \Delta(x) = \Delta_0} \tag{12}$$

which is an inverse of the function

$$W^4(x-y) = \langle T\Phi^2(x)\, \Phi^2(y) \rangle_{connected}, \tag{13}$$

called polarisation (density fluctuaction) propagator in many-body physics. This is an advantage of the EA for the composite operator that $\Gamma^2$, as well as $\Gamma^4$, can be obtained from $\Gamma[\varphi, \Delta]$ through differentiation. In the conventional approach $\Gamma^4$ cannot be derived directly from the EA, but the polarisation propagator $W^4$ has to be calculated and its inverse has to be found.

In Section 2 the 2PPI expansion for the effective action for the composite operator $\Phi^2(x)$ is discussed, the EP and Green's functions $\Gamma^2$ and $\Gamma^4$ are obtained up to five loops for the scalar theory in the space-time of $n$-dimensions. In Section 3 we discuss the method for $n=1$, when the theory has a physical interpretation of a quantum-mechanical anharmonic oscillator. The EP and spectral properties of Green's functions obtained in 2PPI expansion are studied and the resulting energies are compared with the exact spectrum of the AO. Our conclusions are summarised in Section 4.

## 2. The 2PPI Expansion

Verschelde and Coppens [6] represented the EA for the local composite operator in the form

$$\Gamma[\varphi, \Delta] = -S[\varphi] - 6\lambda \int \varphi^2(x)\, \Delta(x)\, d^n x - 3\lambda \int \Delta^2(x)\, d^n x$$

$$+ \tfrac{1}{2} \int \Delta(x)(\Omega^2[\varphi, \Delta] - m^2)\, d^n x + \Gamma^{2\mathrm{PPI}}[\varphi, \Omega^2[\varphi, \Delta]], \qquad (14)$$

where

$$\Omega^2 = m^2 - J_2[\varphi, \Delta] + 12\lambda(\varphi^2 + \Delta). \qquad (15)$$

They have shown that $\Gamma^{2\mathrm{PPI}}[\varphi, \Omega^2[\varphi, \Delta]]$ is a sum of the 2PPI diagrams, defined as those which stay connected after cutting one or two internal lines meeting in the same vertex. The (inverse) propagator is given by

$$G^{-1}(x, y) = (-\partial^2 + \Omega^2(x))\, \delta(x - y), \qquad (16)$$

where the effective mass $\Omega$ is determined as a function of $\Delta$ and $\varphi$ by inverting the equation

$$\frac{\delta \Gamma^{2\mathrm{PPI}}}{\delta \Omega^2(x)} = -\frac{\Delta(x)}{2}, \qquad (17)$$

obtained from equation (6). We have slightly modified the notation of Verschelde and Coppens, by introducing the effective mass $\Omega[\varphi, \Delta]$ and using $\lambda$ not divided by 4!

The conventional $\Gamma[\varphi]$ is given by $\Gamma[\varphi, \Delta_0]$, with $\Delta_0[\varphi]$ determined by the gap equation

$$\Omega^2[\varphi, \Delta] = m^2 + 12\lambda(\Delta + \varphi^2), \tag{18}$$

obtained by setting $J_2 = 0$ in equation (15). Instead of $\Delta_0[\varphi]$ we shall use a self-consistent mass $\Omega_0[\varphi]$, determined from Eq. 17 and 18.

Green's functions in the 2PPI expansion are obtained as functional derivatives of $\Gamma[\varphi, \Delta]$ at $\varphi = \varphi_0$ and $\Delta = \Delta_0$. For simplicity, we consider here the case of unbroken reflection symmetry, taking $\varphi_0 = 0$. The 2-point Green's function for the elementary field (10) is given by

$$\Gamma^2(x-y) = -(-\partial^2 + \Omega^2)\,\delta(x-y) + \left.\frac{\delta^2 \Gamma^{2\mathrm{PPI}}}{\delta\varphi(x)\,\delta\varphi(y)}\right|_{\varphi(x)=0,\ \Omega(x)=\Omega_0}. \tag{19}$$

and its Fourier transform will be denoted by $\Gamma^2(p)$. The 2-point Green's function for the composite field (12) can be represented by

$$\Gamma^4(x-y) = -6\lambda\delta(x-y) - \tfrac{1}{2}\Pi^{-1}(x-y) \tag{20}$$

where

$$\Pi(x-y) = 2\left.\frac{\delta^2 \Gamma^{2\mathrm{PPI}}}{\delta\Omega^2(x)\,\delta\Omega^2(y)}\right|_{\varphi(x)=0,\ \Omega(x)=\Omega_0} \tag{21}$$

is a sum over irreducible polarisation parts. Its Fourier transform is given by

$$\Gamma^4(p) = -6\lambda - \frac{1}{2\Pi(p)}. \tag{22}$$

The EA can be expanded in powers of the Planck constant $h$, which is equivalent to the expansion in the number of loops [1]. As shown by Verschelde and Coppens, in the EA for the local composite operator only the 2PPI diagrams are present. The one-loop approximation, after setting $h = 1$, is given by

$$\begin{aligned}\Gamma_1[\varphi, \Delta] = &-\int [\tfrac{1}{2}\varphi(x)(-\partial^2 + m^2)\,\varphi(x) + \lambda\varphi^4(x)]\,d^n x \\ &+ \tfrac{1}{2}\int d^n x\,\Delta(x)(\Omega^2[\varphi, \Delta] - m^2 - 12\lambda\varphi^2(x) \\ &- 6\lambda\Delta(x) - \tfrac{1}{2}\,Tr\,Ln\,G^{-1}\end{aligned} \tag{23}$$

with the effective mass equation (17) given by

$$\Delta(x) = G(x, x). \tag{24}$$

The conventional EA, obtained as $\Gamma[\varphi, \Delta_0]$ with a self-consistent mass $\Omega_0$, determined by the gap equation

$$\Omega^2(x) - m^2 - 12\lambda\varphi^2(x) - 12\lambda G(x, x) = 0, \tag{25}$$

coincides with the Gaussian EA obtained in other approaches: in the time-dependent Hartree approximation [7], from the EA for the bilocal composite operator [2], and in the optimized expansion [8]. The Gaussian EP [9] at $\phi = 0$ gives the vacuum energy density in the form

$$V_I(0) = I_1(\Omega_0) + \tfrac{1}{2}(m^2 - \Omega_0^2)\, I_0(\Omega_0) + 3\lambda I_0^2(\Omega_0), \tag{26}$$

with $\Omega_0$ fulfiling an algebraic gap equation

$$\Omega^2 - m^2 - 12\lambda I_0(\Omega) = 0, \tag{27}$$

where

$$I_1(\Omega) = \frac{1}{2}\int \frac{d^n p}{(2\pi)^n} \ln(p^2 + \Omega^2) \qquad \text{and} \qquad I_0(\Omega) = \int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2 + \Omega^2}. \tag{28}$$

To this approximation the two-point vertex for the elementary field

$$-\Gamma_1^2(p) = p^2 + \Omega_0^2, \tag{29}$$

is the same as obtained from the Gaussian EA. The 2-point vertex for the composite field, obtained by differentiation of $\Gamma_1[\varphi, \Delta]$, is equal to

$$-\Gamma_1^4(p) = 6\lambda + \frac{1}{I_{-1}(p)} \tag{30}$$

where

$$I_{-1}(p) = 2\int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 + \Omega^2)((p+q)^2 + \Omega^2)}. \tag{31}$$

This is at variance with the inverse of the polarisation propagator, which is obtained [8] from the Gaussian EA in the form

$$W_1^4(p) = \frac{I_{-1}(p)}{1 + 6\lambda I_{-1}(p)} - 288\lambda^2 \times \int \frac{d^n q\, d^n q'}{[1 + 6\lambda I_{-1}(q+q')](q^2 + \Omega^2)[(p+q)^2 + \Omega^2](q'^2 + \Omega^2)[(p+q')^2 + \Omega^2]}. \tag{32}$$

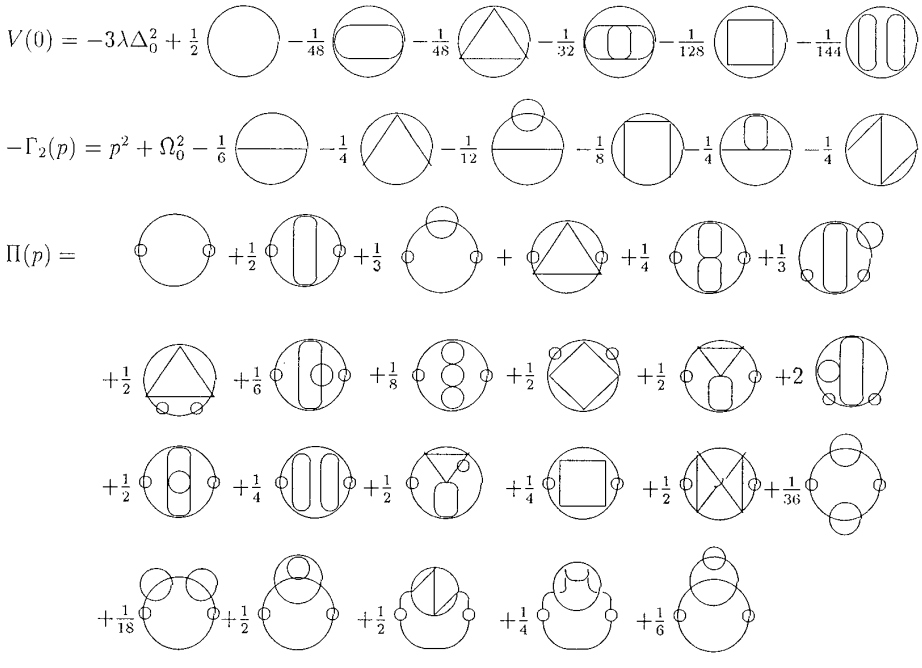

FIG. 1. The value of the EP at $\phi = 0$ and the functions $\Gamma^2(p)$ and $\Pi(p)$ in the 2PPI expansion up to five loops. The small circle denotes an initial or final momentum $P_-$.

Higher orders of the 2PPI expansion provide corrections to the Gaussian approximation. The EP and vertices can be calculated from the given order approximation of $\Gamma[\varphi, \Delta]$. The obtained expressions can be represented in terms of Feynman diagrams in momentum space, since $\phi$, $\Delta_0$ and $\Omega_0$ are space-time independent. The gap equation for $\Omega_0$ becomes an ordinary non-linear equation. In Fig. 1 we show a diagrammatic representation for $V(0)$ and $\Pi(p)$ (up to five loops) and for $\Gamma^2(p)$ (up to four loops). Analytical expressions can be read from the figure.

## 3. Quantum-Mechanical Anharmonic Oscillator

In the space-time of one dimension the $\lambda\Phi^4$ theory is equivalent to the quantum mechanics of the AO with a Hamiltonian given by

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}m^2x^2 + \lambda x^4. \tag{33}$$

The Euclidean propagators are given by

$$W^2(T) = G_c(T, 0) = \sum_1^\infty |\ll 0|\ x\ |k\gg|^2\ e^{-|T|\,\varepsilon_k} \tag{34}$$

and

$$W^4(T) = G_c(T, T, 0, 0) + 2G_c^2(T, 0) = \sum_2^\infty |\ll 0|\ x^2\ |k\gg|^2\ e^{-|T|\,\varepsilon_k}, \tag{35}$$

where $G_c$ are connected Green's functions with the number of points equal to the number of arguments, $|k\gg$ denotes the $k$th excited state of the AO with excitation energy $\varepsilon_k = E_k \times E_0$, and $x$ is a position operator in the Schrödinger representation. The Fourier transform gives the functions

$$W^2(p) = \sum_1^\infty \frac{2\varepsilon_k\ |\ll 0|\ x\ |k\gg|^2}{p^2 + \varepsilon_k^2}; \qquad W^4(p) = \sum_2^\infty \frac{2\varepsilon_k\ |\ll 0|\ x^2\ |k\gg|^2}{p^2 + \varepsilon_k^2} \tag{36}$$

which have poles at imaginary momenta, their absolute values are equal to excitation energies. The Green's functions $\Gamma^2$ and $\Gamma^4$, generated from the exact EA for the local composite operator, are the inverses of the above propagators. They have an infinite number of zeros, which determine odd and even excitations of the AO, respectively. However, in the given order of the 2PPI expansion every Green's function has only a finite number of zeros and gives an approximation of some part of the energy spectrum.

The EP and Green's functions in the 2PPI expansion, read from Fig. 1, can be calculated easily in one dimensional space-time. The powers of $h$ are given explicitely in the obtained expressions to identify orders of the expansion, in the given order approximation we put $h = 1$. The value of the EP at $\phi = 0$ gives the ground state energy equal to

$$E_0 = -\frac{(\Omega_0^2 - m^2)^2}{48\lambda} + h\frac{\Omega_0}{2} - h^3\frac{3\lambda^2}{8\Omega_0^5} + h^4\frac{27\lambda^3}{16\Omega_0^8} - h^5\frac{2373\lambda^4}{128\Omega_0^{11}} \tag{37}$$

with $\Omega_0$ determined by the gap equation

$$\frac{(\Omega^2 - m^2)}{12\lambda} - h\frac{1}{2\Omega} - h^3\frac{15\lambda^2}{8\Omega^7} + h^4\frac{27\lambda^3}{2\Omega^{10}} - h^5\frac{26103\lambda^4}{128\Omega^{13}} = 0. \tag{38}$$

The 2-point vertex for the elementary field, read from Fig. 1, is calculated to be equal to

$$-\Gamma^2(p) = \Omega_0^2 + p^2 - \frac{72h^2\lambda^2}{(9\Omega_0^2 + p^2)\,\Omega_0^2} + \frac{108h^3\lambda^3(45\Omega_0^2 + p^2)}{\Omega_0^5(9\Omega_0^2 + p^2)^2} - \frac{27h^4\lambda^4(589275\Omega_0^6 + 73413\Omega_0^4 p^2 + 2841\Omega_0^2 p^4 + 47p^6)}{\Omega_0^8(9\Omega_0^2 + p^2)^3\,(25\Omega_0^2 + p^2)} \tag{39}$$

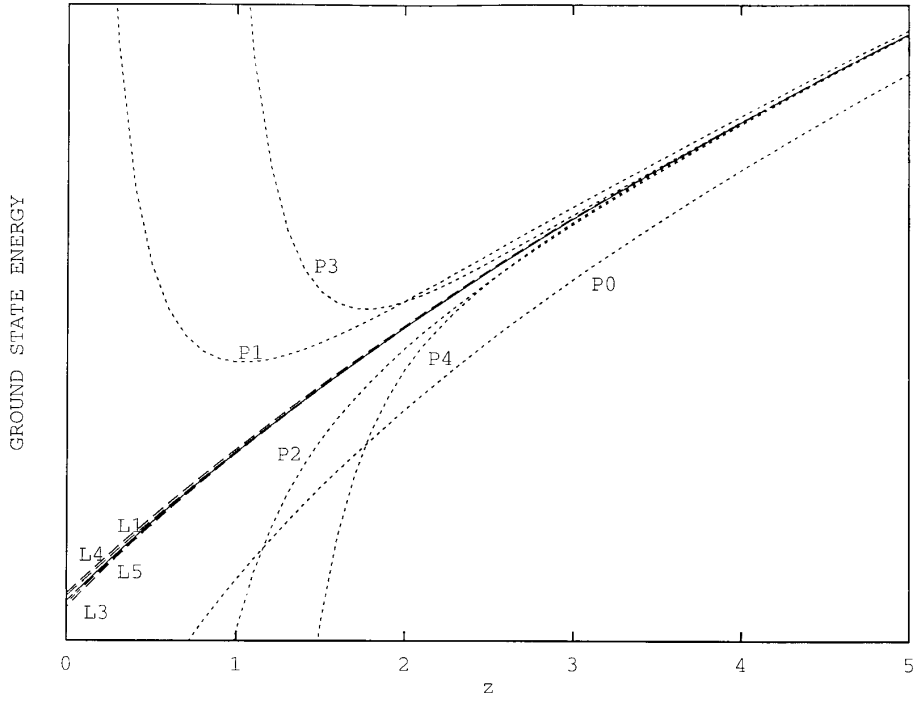


FIG. 2. The ground-state energy of the AO, obtained to the given number of loops in the 2PPI expansion (*dashed lines*), plotted vs. $z = m^2/2\lambda^{2/3}$; compared with the exact value (*solid line*) and given order perturbative results (*dotted lines*).

and $\Gamma^4(p)$ can be found from (22) with the irreducible polarisation given by

$$\begin{aligned}\Pi(p) = &\frac{\hbar}{(4\Omega_0^2+p^2)\,\Omega_0} + \frac{3\hbar^3\lambda^2(2240\Omega_0^4+148\Omega_0^2p^2+5p^4)}{4\Omega_0^7(4\Omega_0^2+p^2)^2\,(16\Omega_0^2+p^2)}\\ &- \frac{27\hbar^4\lambda^3(10240\Omega_0^6+816\Omega_0^4p^2+45\Omega_0^2p^4+p^6)}{\Omega_0^{10}(4\Omega_0^2+p^2)^2\,(16\Omega_0^2+p^2)^2}\\ &+ \frac{3\hbar^5\lambda^4}{64\Omega_0^{13}(4\Omega_0^2+p^2)^3\,(16\Omega_0^2+p^4)^3\,(36\Omega_0^2+p^2)}\\ &\times(266867048448\Omega_0^{12}+97427046400\Omega_0^{10}p^2+11156323328\Omega_0^8p^4\\ &+783135808\Omega_0^6p^6+35476144\Omega_0^4p^8+876428\Omega_0^2p^{10}+8701p^{12}). \quad (40)\end{aligned}$$

Excitation energies are calculated as zeros of $\Gamma$-function, determined to the given order in $\hbar$. The root of $\Gamma^2(p)=0$ is calculated to be at

$$\varepsilon_1 = \Omega_0 - \frac{9\hbar^2\lambda^2}{2\Omega_0^5} + \frac{297\hbar^3\lambda^3}{8\Omega_0^8} - \frac{4599\hbar^4\lambda^4}{8\Omega_0^{11}}, \quad (41)$$

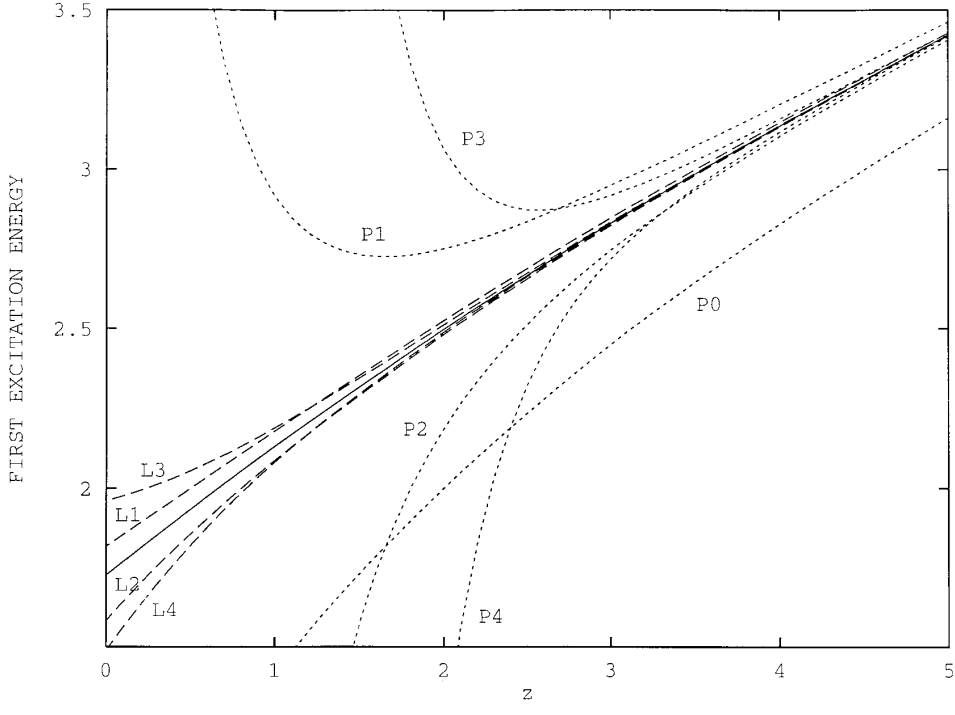


FIG. 3. As in Fig. 2, but for first excitation energy of the AO, obtained as zero of $\Gamma_2$ function in the 2PPI expansion.

and that of $\Gamma^4(p)=0$ is equal to

$$\varepsilon_2 = 2\Omega_0 + \frac{3h\lambda}{\Omega_0^2} - \frac{117h^2\lambda^2}{4\Omega_0^5} + \frac{3159h^3\lambda^3}{8\Omega_0^8} - \frac{488565h^4\lambda^4}{64\Omega_0^{11}}. \tag{42}$$

Expanding the above energies to the fourth order in powers of $\lambda$ gives

$$\begin{aligned}
\varepsilon_0^{pert} &= m/2 + \frac{3\lambda}{4m^2} - \frac{21\lambda^2}{8m^5} + \frac{333\lambda^3}{16m^8} - \frac{30885\lambda^4}{128m^{11}} \\
\varepsilon_1^{pert} &= m + \frac{3\lambda}{m^2} - \frac{18\lambda^2}{m^5} + \frac{1791\lambda^3}{8m^8} - \frac{3825\lambda^4}{m^{11}} \\
\varepsilon_2^{pert} &= 2m + \frac{9\lambda}{m^2} - \frac{297\lambda^2}{4m^5} + \frac{9873\lambda^3}{8m^8} - \frac{1772685\lambda^4}{64m^{11}},
\end{aligned} \tag{43}$$

in agreement with the perturbation theory for Schrödinger equation. The perturbative energies (43) can be obtained from the loop expansion of the conventional EA [8]; however, more diagrams has to be evaluated and a calculation of $\varepsilon_2$ is

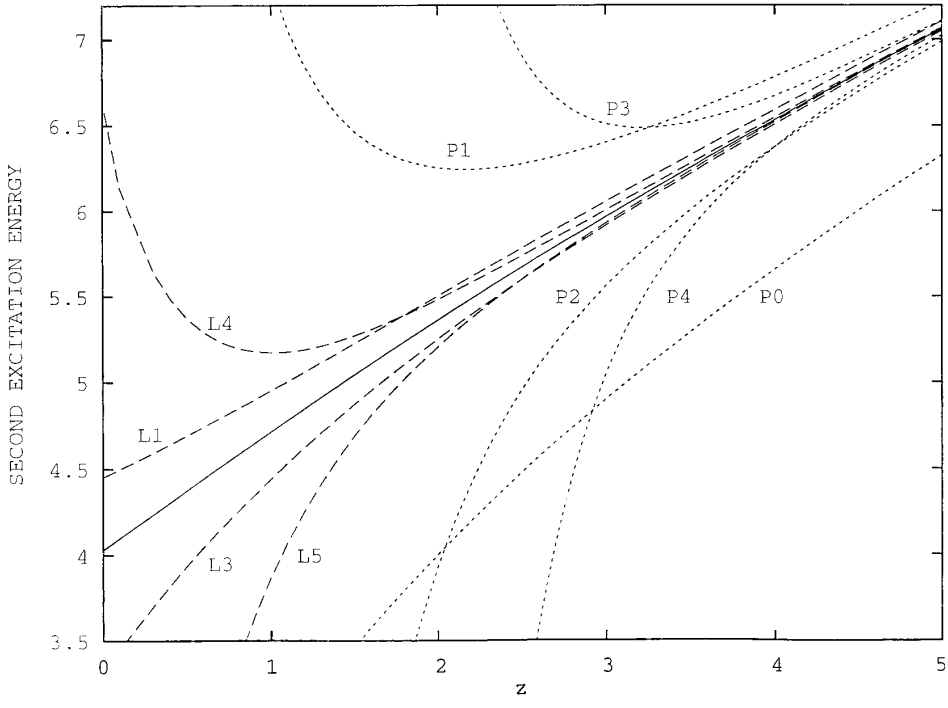


FIG. 4. As in Fig. 2, but for second excitation energy of the AO, obtained as zero of $\Gamma_4$ function in the 2PPI expansion.

furthermore complicated, since $\Gamma^4$ cannot be derived directly from the conventional EA. The 2PPI expansion provides the simplest way for a field theoretical derivation of the perturbative (in $\lambda$) results for the ground state and two lowest excitations.

We have studied numerical results for the ground state energy and two lowest excitations in the 2PPI expansion. To the given order, the largest positive root of the gap equation has been found numerically; in the case where the solution became complex, the real part of the approximant was taken. The results are compared with perturbative energies (43) and exact eigenvalues, calculated by the numerical procedure based on the modification of the linear variational method [10]. All results are presented as functions of a dimensionless fraction $z = m^2/2\lambda^{2/3}$, which is the only parameter of the theory, after rescaling all quantities in terms of $\lambda$.

The results for ground state energy (37) in successive orders of 2PPI expansion are shown in Fig. 2. The quality of the approximation is very good in the whole range of the parameter $z$, only for values smaller than $z_0^{2PPI} \approx 0.2$ small discrepancies between different orders of the 2PPI expansion and the exact result appear. The perturbative results, differ heavily below the much larger critical value $z_0^{pert} \approx 2.5$. In Fig. 3 we show the first excitation energy (41) in successive orders of the 2PPI expansion, compared with the exact result. There is the critical value $z_1^{2PPI} \approx 1.3$,

above which the agreement is good; this value is much smaller than the critical value for perturbative results $z_1^{pert} \approx 3.3$. In Fig. 4 the energies (42), obtained by solving $\Gamma^4(p) = 0$ to the given order in $\hbar$, are compared with the second excitation energy. The agreement is worse than in previous cases. The critical value $z_2^{2PPI} \approx 2.2$ is again smaller than a critical value for perturbative results $z_2^{pert} \approx 4$.

## 4. Conclusions

The 2PPI expansion for EA for the local composite operator is a convenient tool to study a vacuum and lowest excitations in the scalar quantum field theory. The one-loop result gives the Gaussian approximation, where the effective propagator is a Hartree one. Successive approximations of the EP and the Green's functions $\Gamma^2$ and $\Gamma^4$ have been obtained; to each order the effective mass is determined from the algebraic gap equation.

The results for ground state energy and two lowest excitations have been calculated in successive orders of 2PPI expansion for the theory in the space-time of one dimension, i.e., for quantum mechanical AO. The gap equation has been solved numerically. The ground state energy was obtained as a value of the EP at $\phi = 0$ and excitation energies were determined as zeros of appropriate Green's functions. A comparison with the exact spectrum of the AO shows that the convergence of the 2PPI expansion is the best for the ground state. For excited states the critical value above which the expansion converges to the exact result is greater. The higher excitation, the larger critical value, and the region of applicability of the approximation diminishes. This is similar as in the perturbation theory; however, the region of applicability of the 2PPI expansion is much larger for all energy levels.

The numerical results of the 2PPI approach for the lowest excitations up to second order are very similar to that obtained in the optimized expansion and the large N expansion to the same order [8]. Even to this order, the calculations in the later methods are not as straightforward as in the 2PPI expansion, since $\Gamma^4$ cannot be derived directly. In higher orders the optimized and large N expansion would become even more complicated, requiring to solve a gap equation of Bethe-Salpeter type. With the use of the Cornwall-Jackiw-Tomboulis EA for bilocal composite operators the situation would be very much the same. Therefore, the EA for local composite operators provides the simplest method to study the ground state and the lowest excitations in the case of the AO. We hope that the application of the 2PPI expansion to the EA for local composite operators will also appear useful for approximate study of two-particle excitations in quantum field theories in higher dimensional space-time.

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