High energy collisions from nonextensive perspective

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X Polish Workshop on Relativistic Heavy-Ion Collisions
Unreasonable effectiveness of statistical approaches to high-energy collisions
Institute of Physics, Jan Kochanowski University; Kielce, Poland; December 14-15, 2013

Collaborators:

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(*) and:

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- T.Osada , M.Biyajima, M.Kaneyama, T.Mizoguchi, N.Nakamija , N.Suzuki (Japan);
- O.V.Utyuzh, W. Wolak (Poland)

High energy collisions from nonextensive perspective

What does it mean?....

$$\frac{1}{T}\exp\left(-\frac{E}{T}\right)$$

$$\frac{2-q}{T}\left[1-(1-q)\frac{E}{T}\right]^{\frac{1}{1-q}}$$

exp (…) → exp $_q$ (…)

Boltzman-Gibbs statistics → Tsallis statistics

q = nonextensivity parameter

Nonextensivity is phenomenon which is ubiquitous in all branches of science and very well documented. It occurs always whenether:

(*) there are long range correlations in the system (or „system is small” – like our Universe with respect to the gravitational interactions)

(*) there are memory effects of any kind

(*) the phase-space in which system operates is limited or has fractal structure

(*) there are intrinsic fluctuations in the system under consideration

(*) the process proceeds via branching phenomena (in multiplicative manner)

(*) ……………………………

Tsallis distribution

C. Tsallis, J.Stat.Phys. **52** (1988) 479

$$\frac{2-q}{T}\left[1-(1-q)\frac{E}{T}\right]^{\frac{1}{1-q}}$$

$q \to 1$

meaning of q ?

Examples of mechnisms leading to Tsallis distribution:

-q-thermodynamics
- Superstatistics
- Stochastic network approach
- Multiplicative noise
- MaxEnt (Shannon entropy)

more information:
arXiv:1307.7855
AIP1558(2013)893

BG

$$\frac{1}{T}\exp\left(-\frac{E}{T}\right)$$

R. Hagedorn (1965)

BUT…. a bit of history which must be remembered….

First attempts to fit the whole range of p_T are from 1977 (C.Michael) (*):

$$f(p_T) = C\left(1+\frac{p_T}{p_0}\right)^{-n} \to \begin{cases} \exp\left(-\frac{n}{p_0}p_T\right) & \text{for } p_T \to 0 \\ \left(\frac{p_0}{p_T}\right)^{n} & \text{for } p_T \to \infty. \end{cases}$$

→ "soft" (nonperturbative) physics

→ "hard" (perturbative) physics

It is known as „QCD-inspired Hagedorn formula"

(*) C.Michael and L.Vanryckeghen, J.Phys. G3 (1977) L151;
C.Michael, Prog. Part. Nucl. Phys. 2 (1979)1 See also:

Rediscovered as „QCD-inspired formula" (or „Hagedorn distribution") in:
G. Arnison et al. [UA1 Coll.], Phys. Lett. B118, 167 (1982);
R. Hagedorn, Riv. Nuovo Cim. 6 (10), 1 (1983).

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"soft" (nonperturbative) physics

"hard" (perturbative) physics

It is known as „QCD-inspired Hagedorn formula"

NOTICE: for $n = \frac{1}{q-1}$ and $p_0 = \frac{T}{q-1}$ one recovers Tsallis formula.

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C. Tsallis, J.Stat.Phys. **52** (1988) 479

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more information: arXiv:1307.7855 AIP1558(2013)893

?

BG

R. Hagedorn (1

Details can be found , for example, in some our recent works:

(*) **AIP1558(2013)893**; *On Possible Origins of Power-law Distributions*

(*) **PlB727(2013)163** ; *Self-similarity in jet events following from pp collisions at LHC*

(*) **JPG(2012)095004**; *On the possibility of q-scaling in high-energy production processes;*

(*) **APPB34(2012)2047**; *Tsallis fits to* p_T *spectra for pp collisions at LHC;*
PRD 87(2013)114007; *Tsallis fits to pT spectra and multiple hard scattering in pp collisions at the LHC*

(*) **EPJA48(2012)161**; *Consequences of temperature fluctuations in observables measured in high-energy collisions;*
CEJP10(2012)568; *The imprints of superstatistics in multiparticle production* processes**;**
JPG38(2011)065101; *Equivalence of volume and temperature fluctuations in power-law ensembles ;*
EPJA40(2009)299; *Power laws in elementary and heavy-ion collisions*.

… and in earlier references therein ….

It all started from observation of
Long flying component in cosmic rays
[WW, PRD50 (1994) 2318]

(*) observation of deviation from the expected exponential behaviour

(*) successfully intrepreted in terms of cross-section fluctuation:

(*) can be also fitted by:

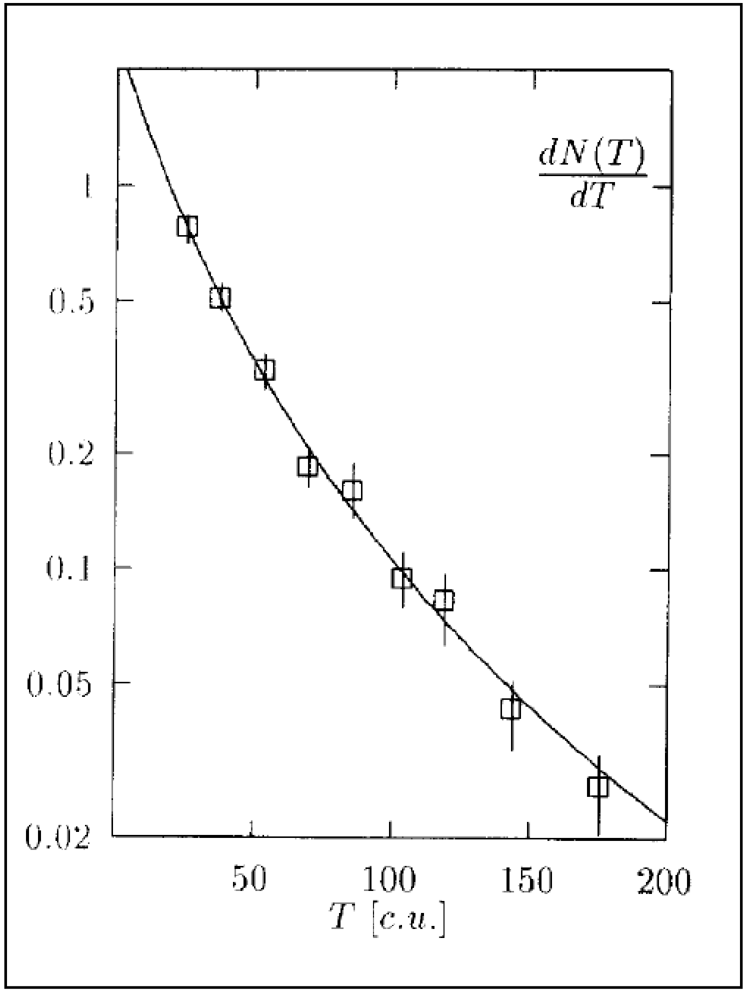
$$\frac{dN}{dT} = \frac{1}{\lambda}\exp\left(-\frac{T}{\lambda}\right) \Rightarrow$$

$$\frac{dN}{dT} = \frac{2-q}{\lambda}\left[1-(1-q)\frac{T}{\lambda}\right]^{\frac{1}{1-q}}$$

$$q = 1.3$$

(*) immediate conjecture:

q ⇦⇨ fluctuations present in the system



Depth distributions of starting points of cascades in Pamir lead chamber Cosmic ray experiment (WW, NPB (Proc.Suppl.) A75 (1999) 191

GW and ZW, PRL 84 (2000) 2770 → q measures intrinsic (nonstatistical) fluctuations in the system

$$h(E/T) = \int_0^\infty f(E/T)\, g(1/T)\, d(1/T)$$

$$f(E) = \frac{1}{T}\exp\left(-\frac{E}{T}\right)$$

BG

$$g(1/T) = \frac{1}{\Gamma\left(\frac{1}{q-1}-s\right)}\frac{T_0}{q-1}\left(\frac{1}{q-1}\frac{T_0}{T}\right)^{\frac{1}{q-1}-1-s}\exp\left(-\frac{1}{q-1}\frac{T_0}{T}\right)$$

gamma distr.

$$h_q(E) = \int_0^\infty f(E)\, g(1/T)\, d(1/T) = \frac{2-q}{T_0}\left[1-(1-q)\frac{E}{T_0}\right]^{\frac{1}{1-q}}$$

Tsallis

$$q = 1 + \frac{Var(T)}{<T>^2}$$

This is know at present as Superstatistics: a superposition of two different statistics relevant to driven nonequilibrium systems with a stationary state and intensive parameter fluctuations [C. Beck et al., Physica A322 (2003) 267]

$$h(E/T) = \int_0^\infty f(E/T)g(1/T)d(1/T)$$

$$f(E) = \frac{1}{T}\exp\left(-\frac{E}{T}\right)$$

BG

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Tsallis

$$q = 1 + \frac{Var(T)}{<T>^2}$$

q-thermodynamics

This will not be the subject of my presentation but this view is reasonable and it was shown that nonextensive-thermodynamics satisfies all demands of the usual thermodynamics applied to systems that posses intrinsic fluctuations, memory effects, are limited and/or nonhomogeneous etc. Cf., for example:

O.J.E.Maroney, PRE89(2009)061141

T.S.Biro, *Is there a temperature?* (Springer 2011)

T.S.Biro et al., JPG37(2010)094027; PRE83(2011)061147;
EPJ Web of Conf. 13 (2011)05004; PLB718 (2012) 125.

J.Cleymans et al., JPG39(2012)025006; EPJA48(2012)160

J.Rożynek, G.Wilk, JPG36(2009)125108; EPJ Web of Conf. 13(2011)0500 2

For those interested in more recent information see:

https://indico.cern.ch/conferenceDisplay.py?confId=285968

Tsallis function

Monday, 13 January 2014 from **09:00** to **11:00** (Europe/Zurich)
at CERN (4-3-006 - TH Conference Room)

Monday, 13 January 2014

09:00 - 09:30 A statistical mechanical view on high energy physics *30'*

The use of the celebrated Boltzmann--Gibbs entropy and statistical mechanics is justified for ergodic-like systems. In contrast, complex systems typically require more powerful theories. We will provide a brief introduction to nonadditive entropies and associated nonextensive statistical mechanics, and then present some recent applications to systems such as high-energy collisions, black holes and others.

Speaker: Constantino Tsallis

09:30 - 10:00 The Tsallis Distribution at the LHC. *30'*

In this talk we review applications of the Tsallis distribution in high energy physics covering p-p, p-Pb and Pb-Pb collisons at ALICE, ATLAS and CMS.
Results from STAR and PHENIX will also be reviewed.
The energy dependence of the three parameters, volume, q and T will be discussed in great detail.

Speaker: Jean Cleymans (University of Cape Town)

Surprisingly Close Tsallis Fits to High Transverse Momentum Hadrons Produced at LHC
- confrontation with pQCD

C.-Y.Wong, G.Wilk:

- *Acta Phys. Polon. B43 (2012) 2047*
- *Phys. Rev. D87 (2013) 114007*
- *arxiv: 1309.7330v1 [hep-ph] – proc.of Low-X 2013*

  The Open Nuclear & Particle Physics Journal, in press

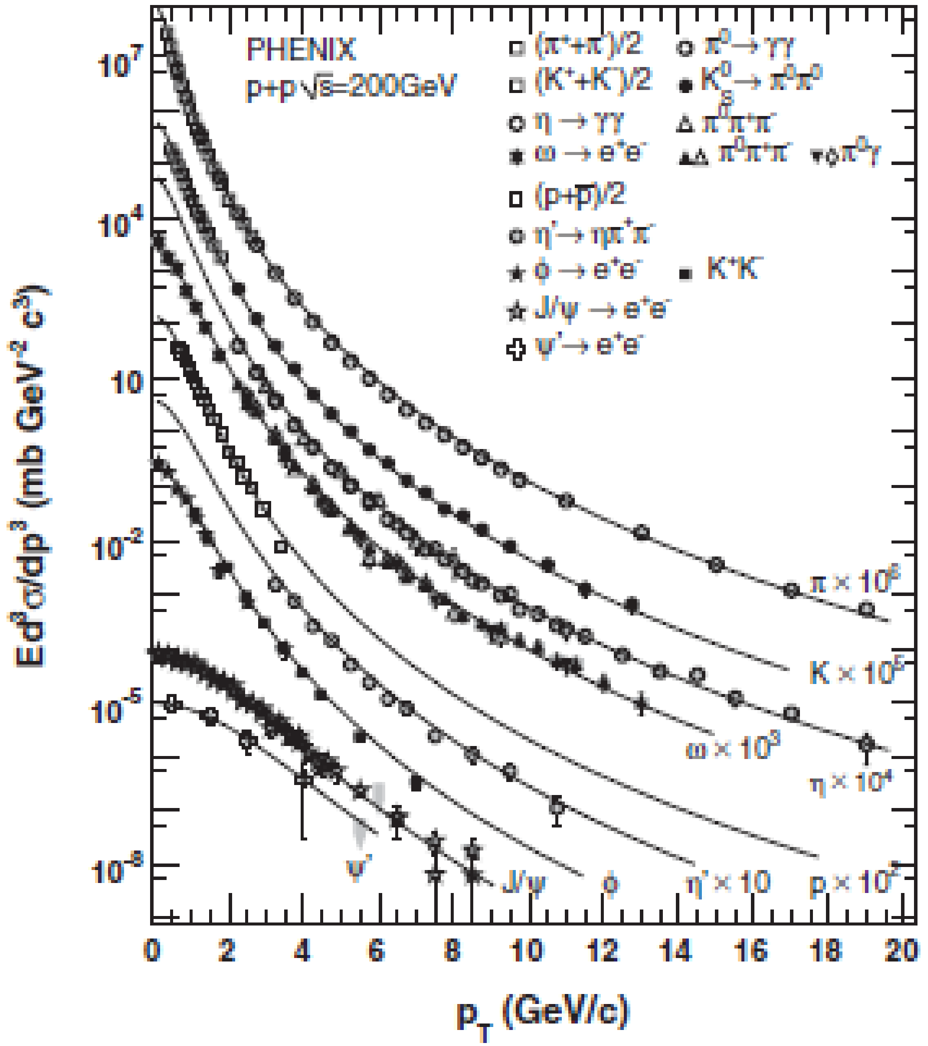
Example of Tsallis distribution: application to PHENIX data

$$E\frac{d\sigma}{d^3p} = \frac{A}{\left(1+\frac{m_T - m}{nT}\right)^n}; \quad n = \frac{1}{q-1}$$

Phenix Coll., PRD 83, 052004 (2011)

Fig. 12
Invariant differential cross sections of different particles measured in p p collisions at √s = 200 GeV in various decay modes.

q=1.1 n=10

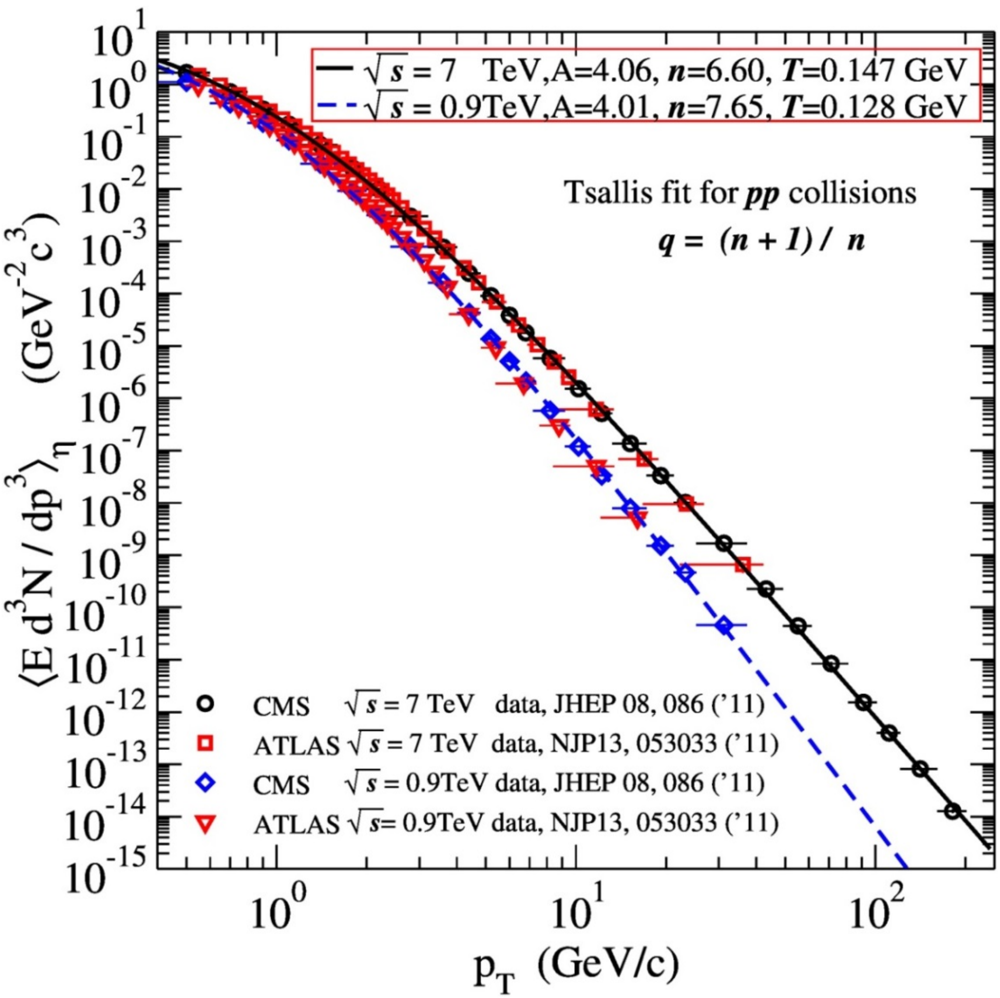


Tsallis distribution can describe LHC p_T distributions

$$E\frac{d\sigma}{d^3p} = \frac{A}{\left(1+\frac{m_T - m}{nT}\right)^n}; \quad n = \frac{1}{q-1}$$

Good Tsallis fits have been obtained

$$\begin{cases} \text{for } \sqrt{s} = 7\text{ TeV}, & n = 6.60 \\ & q = 1.15 \\ \text{for } \sqrt{s} = 0.9\text{ TeV}, & n = 7.65 \\ & q = 1.13 \end{cases}$$



Wong and Wilk, ActaPhysPol.B43,2047(2012)

Tsallis distribution can describe LHC p_T distributions

$$E\frac{d\sigma}{d^3p} = \frac{A}{\left(1+\frac{m_T - m}{nT}\right)^n}; \quad n = \frac{1}{q-1}$$

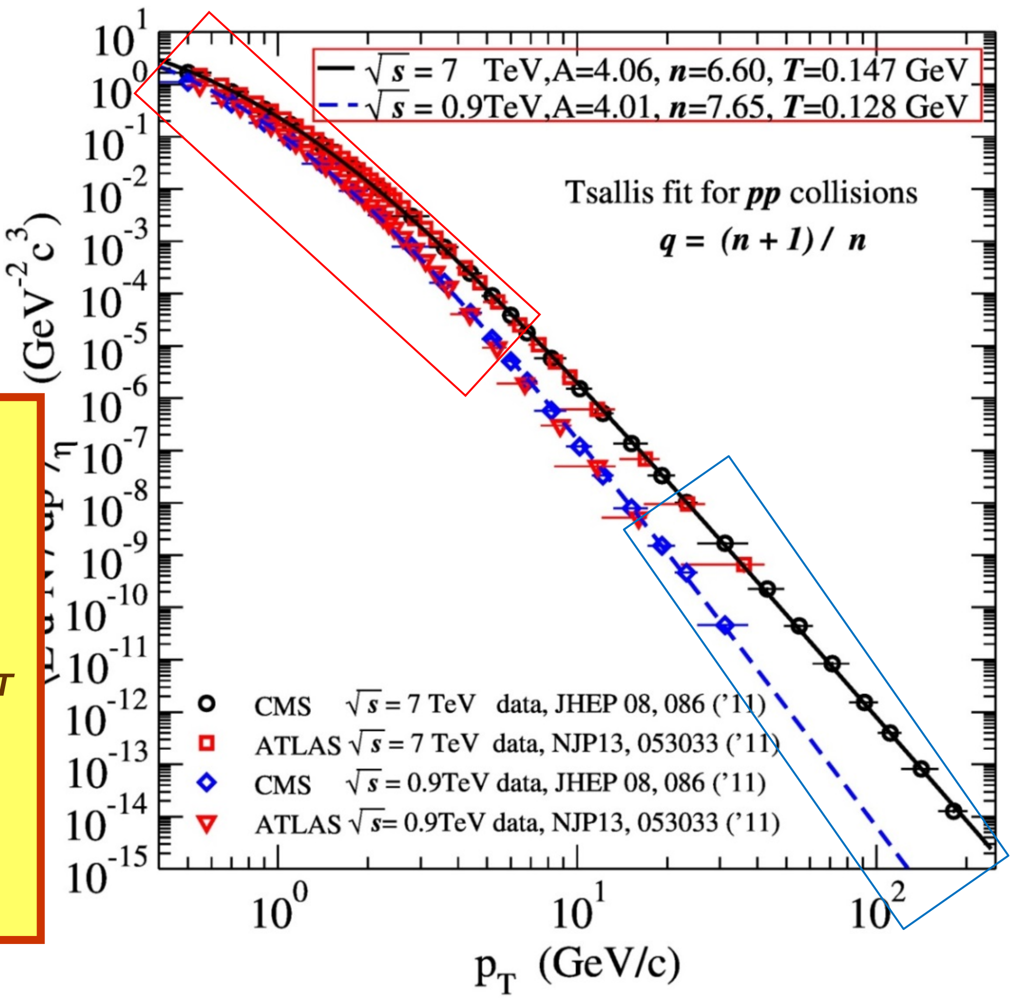
Good Tsallis fits have been

Notice that:

Tsallis fit describes

THE WHOLE RANGE OF VARIABLE p_T

notwithstanding the fact that they are believed to correspond to different dynamics



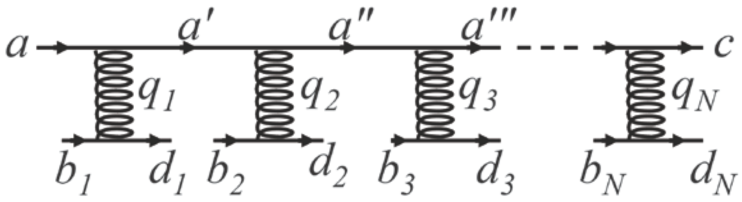
Wong and Wilk, ActaPhysPol.B43,2047(2012)

Good Tsallis p_T fits raise questions

- What is the physical meaning of *n* ?
- If *n* is the power index of $1/p_T^n$, then why is $n \sim 7$, whereas pQCD predicts $n \sim 4$?
- Why are there only few degrees of freedom over such a large p_T domain ?
- Do multiple parton collisions play any role in modifying the power index *n*?
- Does the hard scattering process contribute significantly to the production of low-p_T hadrons?
- What is the origin of low-p_T part of Tsallis fits ?
- ………………………………………………………………………

Parton Multiple Scattering

For the collision of a parton ***a*** with a target of **A** partons in sequence without centrality selection, the differential c_T distribution is given by



$$\frac{d\sigma_H(aA \to cX)}{d\vec{c}_T} = A\frac{\alpha_s^2}{c_T^4}\int d\vec{b}\,T(b)$$

$$+\frac{A(A\text{-}1)}{2}\frac{16\pi\alpha_s^4}{c_T^6}\ln\left(\frac{c_T}{2p_0}\right)\int d\vec{b}[T(b)]^2$$

$$+\frac{A(A\text{-}1)(A-2)}{6}\frac{936\pi^2\alpha_s^4}{c_T^8}\left[\ln\left(\frac{c_T}{2p_0}\right)\right]^2\int d\vec{b}[T(b)]^3$$

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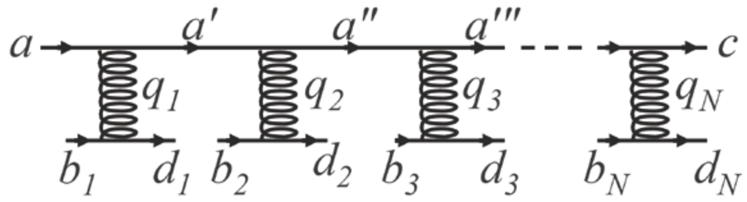

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The contribution from the single collision dominates, but high multiple collisions come in at lower $\mathbf{p_T}$ and for more central collisions

Good Tsallis p_T fits raise questions

- What is the physical meaning of n ?
- If n is the power index of $1/p_T^n$, then why is $n \sim 7$,

We seek answers to these questions from the relativistic hard-scattering (RHS) model

[Blankebecler, Brodsky et al., PRD10(1974)2973; D12(1975)3469; D15(1977)3321]

of low-p_T hadrons ?

- What is the origin of low-p_T part of Tsallis fits ?
- ....

RHS Model - The Power Index in Jet Production

For $gg \to gg, qq' \to qq'$, and $qg \to qg$, $\quad \dfrac{d\sigma(ab \to cd)}{dt} \propto \dfrac{\alpha_s^2(c_T)}{c_T^4}$

The analytical formula is

$$E_c \frac{d\sigma(AB \to cX)}{d^3c} \propto \frac{1}{(c_T/\sqrt{s})^{1/2}} \frac{\alpha_s^2(c_T)}{c_T^4} (1 - x_{a0}(c_T))^g (1 - x_{b0}(c_T))^g$$

For $\eta \sim 0$, $x_{a0}(c_T) = x_{b0}(c_T) = 2x_c(c_T) = 2c_T/\sqrt{s}$, the analytical formula is

$$E_c \frac{d\sigma(AB \to cX)}{d^3c} \propto \frac{\alpha_s^2(c_T)\ (1 - 2x_c(c_T))^g (1 - 2x_c(c_T))^g}{c_T^{4+1/2}/(\sqrt{s})^{1/2}}$$

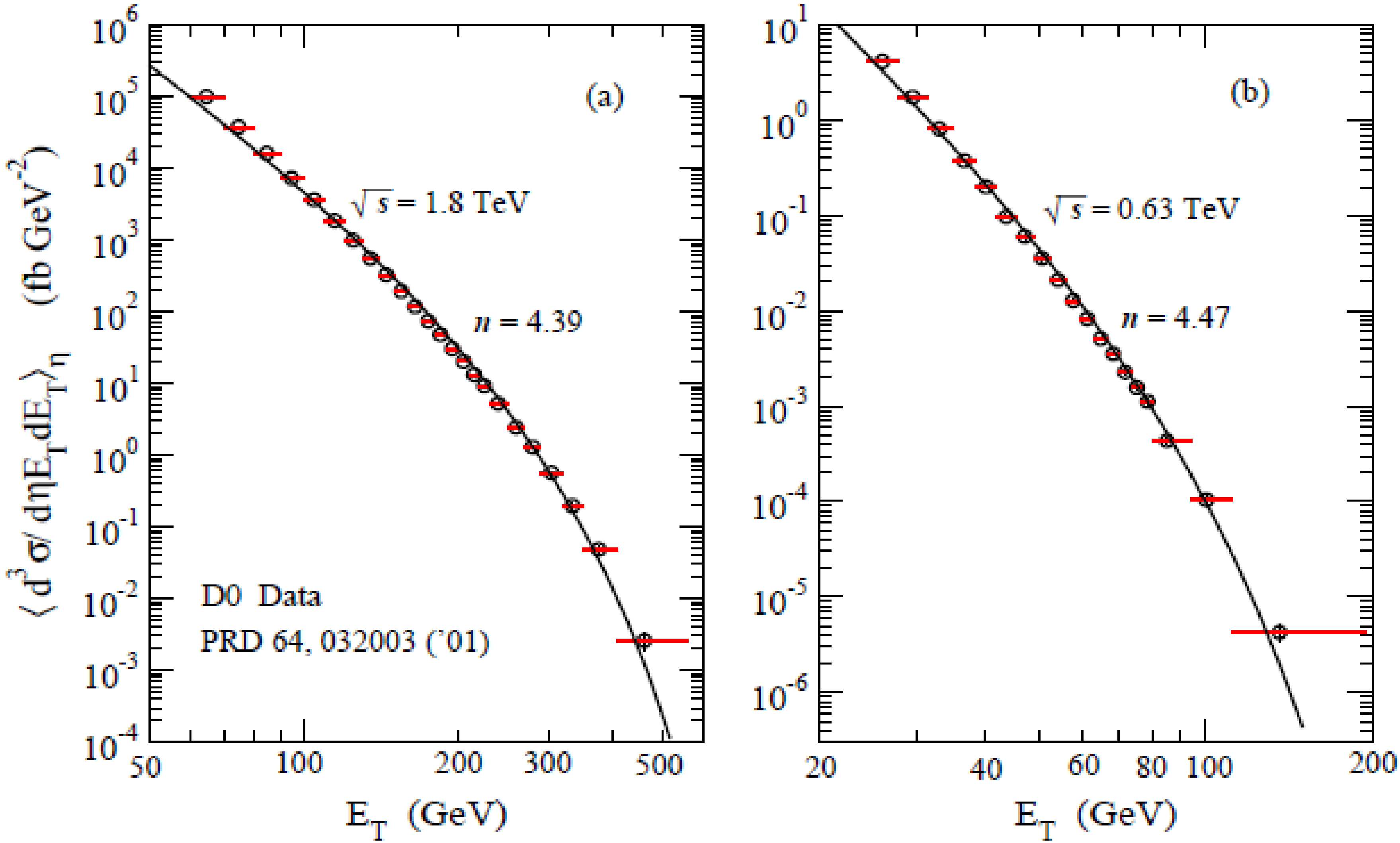
We change notations $\quad c \to p \quad$ and introduce power index n

$$E_p \frac{d\sigma(AB \to pX)}{d^3p} \propto \frac{\alpha_s^2(p_T)\ (1 - 2x_c(p_T))^g (1 - 2x_c(p_T))^g}{p_T^n/(\sqrt{s})^{1/2}} \qquad (19)$$

where $n = 4 + 1/2$ for LO pQCD. $\quad g_{a,b} = g = 6 - 10 \quad$ (we take $g = 6$ [18])

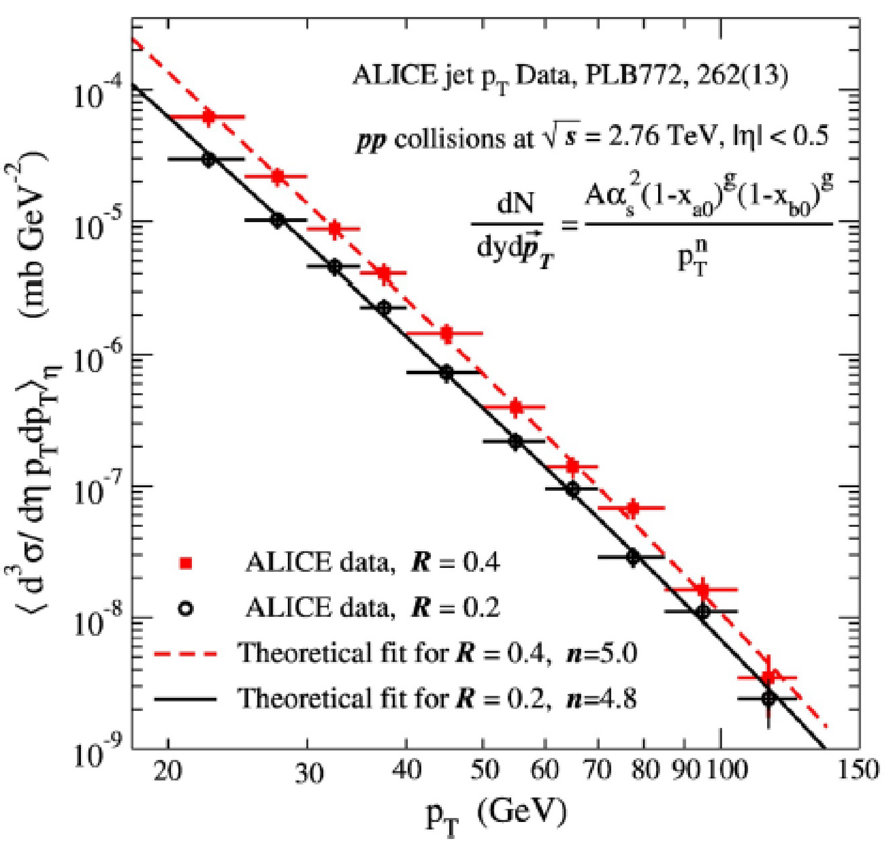
[18] D. W. Duke and J. F. Owens, Phy. Rev D30, 49 (1984).

D0 jet data can be described by RHS model

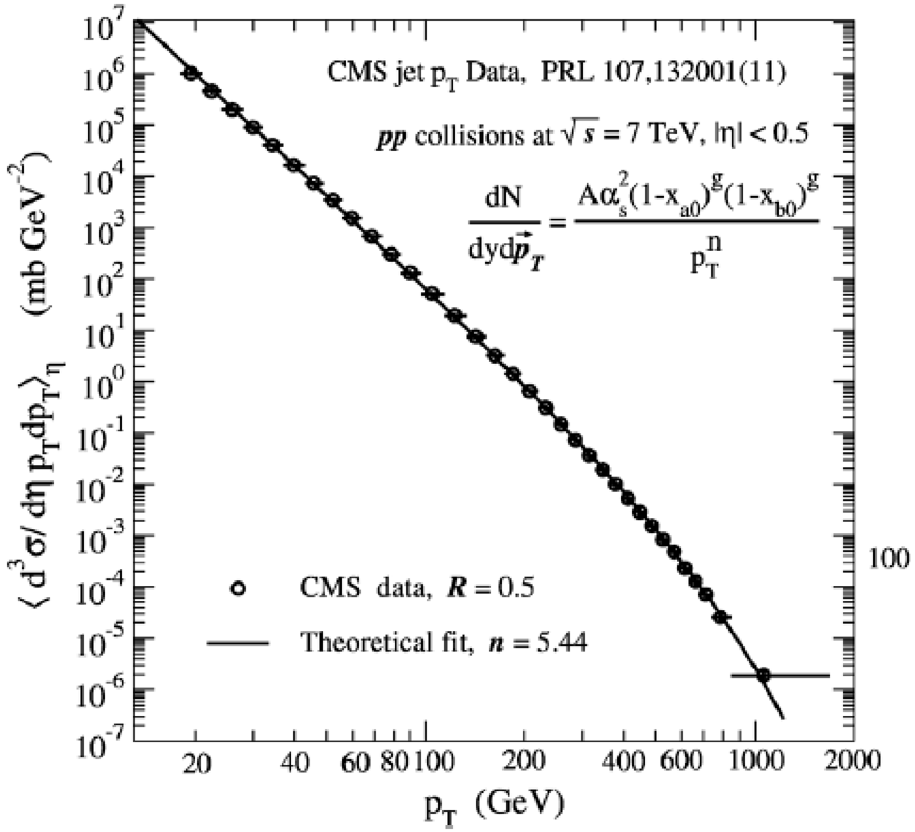


Comparison of the relativistic hard-scattering model results for jet production, Eq. (22) (solid curves), with experimental $d\sigma/d\eta\, E_T\, dE_T$ data from the D0 Collaboration, for hadron jet production within $|\eta|<0.5$, in $\bar{p}p$ collision at (a) $\sqrt{s}$=1.80 TeV, and (b) $\sqrt{s}$=0.63 TeV.

ALICE and CMS jet data can be described by RHS model



n predicted from pQCD is 4.5



n predicted from pQCD is 4.5

(without regularization of $\alpha(p_T)$; R is clustering parameter used in jet finding algorithms)

Except for the CMS data at 7 TeV that may need further re-examination, the power indices extracted for hadron jet production are in approximate agreement with the value of **n=4.5** in Eq. (19) and with previous analysis in [10], indicating the approximate validity of the hard scattering model for jet production in hadron-hadron collisions, with the predominant $\boldsymbol{\alpha_s^2 / c_T^4}$ parton-parton differential cross section as predicted by pQCD.

Collaboration	$\sqrt{s}$	R	η	n
D0	$\bar{p}p$ at 1.80 TeV	0.7	$\lvert\eta\rvert < 0.7$	4.39
D0	$\bar{p}p$ at 0.63 TeV	0.7	$\lvert\eta\rvert < 0.7$	4.47
ALICE	pp at 2.76 TeV	0.2	$\lvert\eta\rvert < 0.5$	4.78
ALICE	pp at 2.76 TeV	0.4	$\lvert\eta\rvert < 0.5$	4.98
CMS	pp at 7 TeV	0.5	$\lvert\eta\rvert < 0.5$	5.39

[10] F. Arleo, S. Brodsky, D. S. Hwang, and A. M. Sickles, Phys. Rev. Lett. 105, 062002 (2010).

Evolution from jet to hadrons

The evolution from a jet to hadrons passes through the stages of

(i) showering (and/or fragmentation),

(ii) hadronization.

Phenomenological Modifications for Hadron Production

Jet production →

$$E_c \frac{d\sigma^3(AB \to cX)}{dc^3} = \frac{A\alpha_s^2(Q^2(c_T))\ (1-x_{a0})^{g_a+\frac{1}{2}}(1-x_{b0})^{g_b+\frac{1}{2}}}{c_T^n\sqrt{1-x_c}}$$

hadrons

(*) For the case of hadron production, it is necessary to take into account additional effects. Jets undergo fragmentation and hadronization to produce the observed hadrons.

(*) For example: from the fragmentation function for a parent parton jet to fragment into hadrons, an observed hadron **p** of transverse momentum p_T can be estimated to arise (on the average) from the fragmentation of a parent jet **c** with transverse momentum $\langle c_T \rangle = 2.3 p_T$ [11].

[11] C. Y. Wong and G. Wilk, Phys. Rev. D87, 114007 (2013).

Effects of showering (and/or fragmentation) on power law

For linear fragmentation where p=zc:

$$E_p \frac{d\sigma(AB\to pX)}{d^3p} = \int dz D_{p/c}(z) \int d^4c \frac{d\sigma(AB\to cX)}{d^4c}\delta^{(4)}(p-zc)$$

$$= \int \frac{dz}{z^2} D_{p/c}(z) z^{4+1/2} \frac{\alpha_s^2(c_T)(1-x_c(c_T))^g(1-x_c(c_T))^g}{p_T^{4+1/2}}$$

$$\approx \frac{\alpha_s^2(\bar{c}_T)(1-x_c(\bar{c}_T))^g(1-x_c(\bar{c}_T))^g}{p_T^{4+1/2}}$$

where $\bar{c}_T = p_T \left\langle \frac{1}{z} \right\rangle$

and $\left\langle \frac{1}{z} \right\rangle = \int \frac{dz}{z^2} D_{p/c}(z) z^{4+1/2} \left(\frac{1}{z} \right) \Big/ \int \frac{dz}{z^2} D_{p/c}(z) z^{4+1/2}$

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where $\bar{c}_T = p_T \left\langle \frac{1}{z} \right\rangle$

and $\left\langle \frac{1}{z} \right\rangle = \int \frac{dz}{z^2} D_{p/c}(z) z^{4+1/2} \left(\frac{1}{z}\right) \Big/ \int \frac{dz}{z^2} D_{p/c}(z) z^{4+1/2}$

The power law and power index are preserved under p=zc fragmentation

Effects of showering (and/or fragmentation) on power law

However, as a result of parton showering involving virtuality degradation, the leading hadron momentum *p* and the showering parton momentum c may not be linearly related and one can expect that

$$\mathbf{p = z\, c^{1-\mu}}$$

where parameter μ describes details of virtuality degradation. As a consequence, the power index can be changed under parton showering.

After the fragmentation and showering of the parton c to hadron p, the hard-scattering cross section for the scattering in terms of hadron momentum p_T becomes

$$\frac{d^3\sigma(AB\to pX)}{dy\,d\mathbf{p}_T}=\frac{d^3\sigma(AB\to cX)}{dy\,d\bar{\mathbf{c}}_\mathbf{T}}\frac{d\bar{\mathbf{c}}_\mathbf{T}}{d\mathbf{p}_T}$$

$$\propto\frac{\alpha_S^2(\bar{\mathbf{c}}_\mathbf{T})(1-x_{a0}(\bar{\mathbf{c}}_\mathbf{T}))^{g_a}(1-x_{b0}(\bar{\mathbf{c}}_\mathbf{T}))^{g_b}}{c_T^{4+1/2}}\frac{d\bar{\mathbf{c}}_\mathbf{T}}{d\mathbf{p}_T}$$

where $$\frac{d\bar{\mathbf{c}}_\mathbf{T}}{d\mathbf{p}_T}=\frac{1}{1-\mu}\left(a\frac{p_T}{p_{T0}}\right)^{\frac{2\mu}{1-\mu}}$$

Here a is a constant relating the scales of virtuality and transverse momentum.

After the fragmentation and showering of the parton c to hadron p, the hard-scattering cross section for the scattering in terms of hadron momentum p_T becomes

$$\frac{d^3\sigma(AB\to pX)}{dyd\mathbf{p}_T}=\frac{d^3\sigma(AB\to cX)}{dyd\bar{\mathbf{c}}_\mathbf{T}}\frac{d\bar{\mathbf{c}}_\mathbf{T}}{d\mathbf{p}_T}$$

$$\propto\frac{\alpha_S^2(\bar{\mathbf{c}}_\mathbf{T})(1-x_{a0}(\bar{\mathbf{c}}_\mathbf{T}))^{g_a}(1-x_{b0}(\bar{\mathbf{c}}_\mathbf{T}))^{g_a}}{c_T^{4+1/2}}\frac{d\bar{\mathbf{c}}_\mathbf{T}}{d\mathbf{p}_T}$$

where $$\frac{d\bar{\mathbf{c}}_\mathbf{T}}{d\mathbf{p}_T}=\frac{1}{1-\mu}\left(a\frac{p_T}{p_{T0}}\right)^{\frac{2\mu}{1-\mu}}$$

Here a is a constant relating the scales of virtuality and transverse momentum. Therefore under the fragmentation $c\to p$, the hard scattering cross section for $AB\to pX$ becomes:

$$\frac{d^3\sigma(AB\to pX)}{dyd\mathbf{p}_T}\propto\frac{\alpha_S^2(\bar{\mathbf{c}}_\mathbf{T})(1-x_{a0}(\bar{\mathbf{c}}_\mathbf{T}))^{g_a}(1-x_{b0}(\bar{\mathbf{c}}_\mathbf{T}))^{g_a}}{p_T^{n'}}$$

$$n'=\left(\frac{n-2\mu}{1-\mu}\right)\quad\text{with}\quad \mathrm{n}=4+1/2\quad\Rightarrow\quad\mu=\left(\frac{n'-n}{n'-2}\right)$$

After all this one gets power-law behavior:

$$\frac{d^3\sigma\left(AB \to pX\right)}{dyd\mathbf{p}_T} \propto \frac{\alpha_S^2\left(\overline{\mathbf{c}}_\mathbf{T}\right)\left(1 - x_{a0}\left(\overline{\mathbf{c}}_\mathbf{T}\right)\right)^{g_a}\left(1 - x_{b0}\left(\overline{\mathbf{c}}_\mathbf{T}\right)\right)^{g_a}}{p_T^{n'}} \qquad (*)$$

but **it is not Tsallis formula - low p_T behavior is not correct.**

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but **it is not Tsallis formula - low p_T behavior is not correct.**

The proposed possible remedy is to replace the usual parameter p_0 ($\sim 1 \div 2$ GeV) dividing phase space into part governed by „soft physics „ ($p_T < p_0$) from that governed by „hard physics" ($p_T \geq p_0$) by regularizing denominator in (*), for example by using:

$$\frac{d^3\sigma(AB\rightarrow pX)}{dyd\mathbf{p}_T}\propto\frac{\alpha_S^2(\bar{\mathbf{c}}_\mathbf{T})(1-x_{a0}(\bar{\mathbf{c}}_\mathbf{T}))^{g_a}(1-x_{b0}(\bar{\mathbf{c}}_\mathbf{T}))^{g_a}}{\left(1+\frac{m_T}{m_{T0}}\right)^n} \qquad (\oplus)$$

$$m_T=\sqrt{m^2+p_T^2}$$

After all this one gets power-law behavior:

$$\frac{d^3\sigma(AB \to pX)}{dyd\mathbf{p}_T} \propto \frac{\alpha_S^2(\bar{\mathbf{c}}_\mathbf{T})\left(1-x_{a0}(\bar{\mathbf{c}}_\mathbf{T})\right)^{g_a}\left(1-x_{b0}(\bar{\mathbf{c}}_\mathbf{T})\right)^{g_a}}{p_T^{n'}} \qquad (*)$$

Notice that we have just used the form of Tsallis („Hagedorn") formula showed at the beginning:

$$\mathbf{E}\frac{\mathbf{d\sigma}}{\mathbf{d^3p}} = \frac{\mathbf{A}}{\left(\mathbf{1}+\frac{\mathbf{m_T}-\mathbf{m}}{\mathbf{nT}}\right)^{\mathbf{n}}}$$

$$\frac{d^3\sigma(AB \to pX)}{dyd\mathbf{p}_T} \propto \frac{\alpha_S^2(\bar{\mathbf{c}}_\mathbf{T})\left(1-x_{a0}(\bar{\mathbf{c}}_\mathbf{T})\right)^{g_a}\left(1-x_{b0}(\bar{\mathbf{c}}_\mathbf{T})\right)^{g_a}}{\left(1+\frac{m_T}{m_{T0}}\right)^n} \qquad (\oplus)$$

$$m_T = \sqrt{m^2 + p_T^2}$$

In fact, in actual calculation, in addition to the replacement

$$(1/p_T)^n \rightarrow \quad 1/(1 + p_T/p_0)^n$$

we also regularize in similar manner the coupling constant for small values of p_T

(following method proposed in hadron spectroscopic studies by C. Y. Wong, E. S. Swanson, and T. Barnes, Phy. Rev. C, 65, 014903 (2001)):

$$\boldsymbol{\alpha_s(p_T) = \frac{12\pi}{27\ln(C + p_T^2/\Lambda_{QCD}^2)}},$$

$$\Lambda_{\mathrm{QCD}} = 0.25\,\mathrm{GeV} \Rightarrow \alpha_S\left(M_Z^2\right) = 0.1184;$$

$$C = 10 \Rightarrow \alpha_S\left(Q \propto \Lambda_{QCD}\right) \approx 0.6 \text{ in hadron spectroscopy studies}$$

Experiments measure the differential yield in nonsingle-diffractive events, which in our case is

$$\frac{d^3N\left(AB\to pX\right)}{dyd\mathbf{p}_T}=\sqrt{1-\frac{m^2}{m_T^2\cosh^2 y}}\,A\frac{\alpha_S^2\left(\bar{\mathbf{c}}_\mathbf{T}\right)\left(1-x_{a0}\left(\bar{\mathbf{c}}_\mathbf{T}\right)\right)^{g_a}\left(1-x_{b0}\left(\bar{\mathbf{c}}_\mathbf{T}\right)\right)^{g_a}}{\left(1+\frac{m_T}{m_{T0}}\right)^n}=$$

$$=\sqrt{1-\frac{m^2}{m_T^2\cosh^2 y}}\,A\alpha_S^2\left(\bar{\mathbf{c}}_\mathbf{T}\right)\left(1-x_{a0}\left(\bar{\mathbf{c}}_\mathbf{T}\right)\right)^{g_a}\left(1-x_{b0}\left(\bar{\mathbf{c}}_\mathbf{T}\right)\right)^{g_a}\cdot$$

$$\bullet\left[1-\left(1-q\right)\frac{m_T}{T}\right]^{\frac{1}{1-q}}$$

where $\mathrm{q}=1+\frac{1}{\mathrm{n}}$ and $T=\frac{m_{T0}}{q-1}$

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where $\quad q = 1+\frac{1}{n} \quad$ and $\quad T = \frac{m_{T0}}{q-1}$

Looks like Tsallis

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$$=\sqrt{1-\frac{m^2}{m_T^2\cosh^2 y}}\,A\,\alpha_S^2(\bar{\mathbf{c}}_\mathbf{T})(1-x_{a0}(\bar{\mathbf{c}}_\mathbf{T}))^{g_a}(1-x_{b0}(\bar{\mathbf{c}}_\mathbf{T}))^{g_a}\,\cdot$$

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... where

where $\mathrm{q}=1+\frac{1}{\mathrm{n}}$ and $T=\frac{m_{T0}}{q-1}$

... but with quite complicated prefactor

Looks like Tsallis...

Analysis of hadron p_T distributions

Two ways to regulate the cross sections at low p_T were used:

I. Linear m_T: $\quad p_T \to (m_{T0} + m_T)$

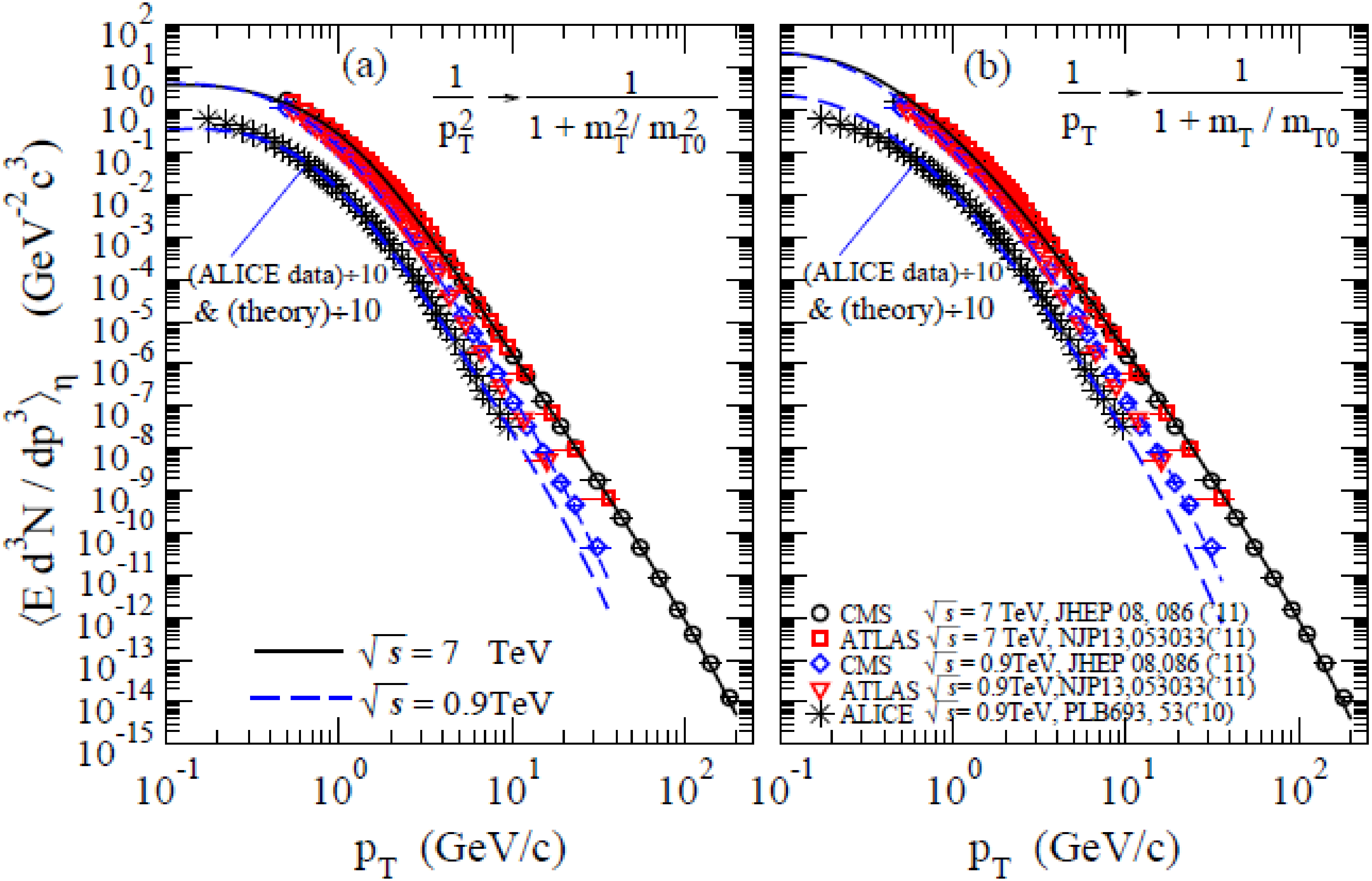
$$E_p \frac{d\sigma(AB \to pX)}{d^3p} = \frac{A\alpha_s^2(\bar{c}_T)\ (1-2x_c(\bar{c}_T))^g (1-2x_c(\bar{c}_T))^g}{(m_{T0}+m_T)^n}. \qquad (24)$$

II. Quadratic m_T: $\quad p_T^2 \to (m_{T0}^2 + m_T^2)$

$$E_p \frac{d\sigma(AB \to pX)}{d^3p} = \frac{A\alpha_s^2(\bar{c}_T)\ (1-2x_c(\bar{c}_T))^g (1-2x_c(\bar{c}_T))^g}{(m_{T0}^2+m_T^2)^{n/2}}. \qquad (25)$$

where $\quad \bar{c}_T = p_T \left\langle \frac{1}{z} \right\rangle; \quad \left\langle \frac{1}{z} \right\rangle = \int \frac{dz}{z^2} D_{p/c}(z) z^{4+1/2} \left(\frac{1}{z} \right) \Big/ \int \frac{dz}{z^2} D_{p/c}(z) z^{4+1/2} = 2.33$

We search for n that fits the hadron p_T spectra.



Comparison of the experimental data for hadron production in pp collisions at the LHC with the relativistic hard scattering model results (solid and dashed curves) (a) using Eq. (25), with a quadratic m_T dependence of the regulating function, and (b) using Eq. (24), with a linear m_T dependence of the regulating function. In both cases regularized coupling constant α_S was used.

	Linear m_T Eq. (24)		Quadratic m_T^2 Eq. (25)	
	$\sqrt{s}$=7TeV	$\sqrt{s}$=0.9TeV	$\sqrt{s}$=7TeV	$\sqrt{s}$=0.9TeV
n	5.69	5.86	5.45	5.49
m_{T0} (GeV)	0.804	0.634	1.09	0.837

(*) For pp collisions at the LHC the power index extracted from hadron spectra has the value of n~6 and is slightly greater than the power indices of n~4-5 extracted from jet transverse differential cross sections.

(*) Fragmentation and showering processes increase therefore slightly the value of the power index n of the transverse spectra.

Conclusions:

- A simple Tsallis formula can describe data with a power index of **n ~ 6.6 - 7.6**

- A power law with a power index of **n ~ 4 - 5** can describe the p_T spectra of jets.

- A regularized power law with a power index of **n ~ 5.5 - 6** can describe (together with regularized coupling constant) the p_T spectra of hadrons for all p_T .

- The power index ***n*** becomes larger as a jet evolves into hadrons.

Tsallis distribution from Shannon entropy

Example: Shannon entropy $S = - \int f(x) \ln[f(x)] dx$

leads to:

(*) exponential for condition $\langle x \rangle = const$

(*) Gauss distribution for condition $\langle x^2 \rangle = const$

(*) gamma distribution for $\langle \ln(x) \rangle = const$

(*) Cauchy distribution for $\langle \ln(1+x^2) \rangle = const$

....

In general: the maximum entropy density for **f(x)** satisfying constraint

$$\int f(x) h(x) dx = const,$$

where **h(x)** is some function of **x**, is of the form

$$f(x) = \exp[\lambda_0 + \lambda h(x)].$$

The constants λ_0 and λ are chosen so that f(x) is normalized and satisfies the constraint.

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E.Rufeil Fiori, A.Plastino,; *A Shannon-Tsallis transformation*, Physica A392, 1742 (2013):

S.Presse, K.Ghosh, J.Lee, K..A. Dill; *Nonadditive Entropies Yield Probability Distributions with Biases notWarranted by the Data*; PRL 111, 180604 (2013)

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Important question for practitioners using Tsallis or Shannon approach to describe their respective data/results: does it mean that they are simply equivalent?

S.Presse, K.Ghosh, J.Lee, K..A. Dill; *Nonadditive Entropies Yield Probability Distributions with Biases notWarranted by the Data*; PRL 111, 180604 (2013)

Tsallis distribution from Shannon entropy

In particular, condition: $\langle z \rangle = z_0 = (q-1)/(2-q)$

where $z = \ln[1 - (1-q)E/T_0]$

results in $f(z) = (1/z_0)\exp(-z/z_0)$

i.e., in Tsallis distribution:

$$f(E) = \frac{1}{(1+z_0)T_0}\left(1 + \frac{z_0}{1+z_0}\frac{E}{T_0}\right)^{\frac{1+z_0}{z_0}}$$

$$= \frac{2-q}{T_0}\left[1-(1-q)\frac{E}{T_0}\right]^{\frac{1}{1-q}}$$

Tsallis distribution from Shannon entropy

Interesting: condition imposed by

$$\langle z \rangle = z_0 = \frac{q-1}{2-q} \quad \text{where} \quad z = \ln\left[1-(1-q)\frac{x}{x_0}\right]$$

is natural for the multiplicative noise described by

$$\frac{dp}{dt} + \gamma(t)p = \xi(t)$$

There is connection between the kind of noise in this process and the condition imposed in the MaxEnt approach:

(*) For processes described by additive noise, $dx/dt = \xi(t)$,
one has **exponential distributions**. The natural condition for them is that imposed on **the arithmetic mean,** $\langle x \rangle = c + E(\xi)t$.

(*) For the multiplicative noise, $dx/dt = x\gamma(t)$,
one has a **power law distribution** for which the natural condition is that imposed on **the geometric mean ,** $\langle \ln x \rangle = c + E(\gamma)\tau$.

Tsallis distribution from Shannon entropy

For interested: more on this subject can be learned from

(*) **http://arxiv.org/abs/cond-mat/0507414v1**
A.Rostovtsev, *On a geometric mean and power-law statistical distributions.*

(*) **http://itia.ntua.gr/1127**
S.M. Papalexiou and D. Koutsoyiannis ,
Entropy maximization, p-moments and power-type distributions in nature

Summary

(*) Tsallis distribution (especially in the form of „QCD-inspired Hagedorn distribution") can be - at least - regarded as the handy (two parameter) parametrization of data..

(*) But it can be also associated

(i) with the existence of some intrinsic fluctuations in the hadronizing system;

(ii) with fact that in reality we predominantly deal with systems of interacting, not free, particles but still try to impose on them BG statistics (distribution);

(iii) with fact that our thermal models assume homegeneous, infinite etc. heat baths, which are not found in most of the hadronizing systems we consider;

(iv) with the existence of different scales, branching phenomena , multiplicative subprocesses – all leading to power rather than exponential distributions (the best representation of which for all values of relevant variables is Tsallis distribution).

Thank you

Information entropy

- Information entropy is a purely probabilistic concept and is regarded as a measure of the uncertainty related to a random variable (RV).
- In literature there are more than twenty different entropy measures [1], proposed mainly as generalizations of Boltzmann-Gibbs-Shannon (BGS) entropy, which is the most famous and well justified entropy measure. The BGS entropy for a non-negative continuous RV X with density function $f_X(x)$ is defined as

$$S_{\mathrm{BGS}} = -\int_0^\infty f_X(x) \ln f_X(x) \mathrm{d}x \qquad (1)$$

- A famous generalization, proposed by Rényi in 1961, is defined as

$$S_{\mathrm{R}} = \frac{1}{1-q} \ln \int_0^\infty f_X(x)^q \, \mathrm{d}x \qquad (2)$$

- Another popular generalization, the Havrda-Charvat-Tsallis (HTC) entropy [2,3], is defined as

$$S_{\mathrm{HCT}} = \frac{1}{1-q}\left[\int_0^\infty f_X(x)^q \mathrm{d}x - 1\right] \qquad (3)$$

- For $q \to 1$ both the Renyi and HCT entropies converge to the BGS entropy.

Jaynes: *finding the maximum entropy under given constraints ->*
the resulying probability distribution "is the least biased estimate possible on the given information…".

- Mathematically, the given information used in the principle of maximum entropy, is expressed as a set of constraints formed as expectations of functions $g_j(\,)$ of X, i.e.,

$$E\left[g_j(x)\right] = \int_0^{\infty} g_j(x)\, f_X(x)\mathrm{d}x = c_j, \quad j = 1, \dots, n \qquad (4)$$

- The resulting maximum entropy distributions emerge by maximizing the selected form of entropy with constraints c_j, and with the additional constraint (to guarantee the legitimacy of the distribution)

$$\int_0^{\infty} f_X(x)\,\mathrm{d}x = 1 \qquad (5)$$

- The general solution of the maximum entropy distributions resulting from the maximization of BGS entropy and the HCT entropy (accomplished by using the method of Lagrange multipliers) are, respectively,

$$f_X(x) = \exp[-\lambda_0 - \textstyle\sum_{j=1}^{n} \lambda_j\, g_j(x)] \qquad (6)$$

$$f_X(x) = \left\{1 + (1-q)\left[\lambda_0 + \textstyle\sum_{j=1}^{n} \lambda_j\, g_j(x)\right]\right\}^{-1/(1-q)} \qquad (7)$$

where λ_j, with j =0,…, n are the Lagrange multipliers linked to the constraints.

Generalized power function and p-moments:

- Here, we generalize the important notion of moments inspired by the limiting definition of the exponential function. We first define the generalized power function

$$x_p^q = \ln(1 + p\,x^q)/p \qquad (8)$$

which for $p \to 0$ becomes the familiar power function x^q. Thus, we can define a generalization of the classical moments, which we name p-moments of order q as

$$m_q^p = E(X_p^q) = \frac{1}{p}\int_0^{\infty} \ln(1 + p\,x^q)\; f_X(x)\mathrm{d}x \qquad (9)$$

Clearly, for $p \to 0$, p-moments are identical to classical moments, i.e., $m_q^0 = m_q \equiv E(X^q)$.

Rationale:

(*) Generalized entropy measures have been successfully used; why not *p*-moments with the standard definition of entropy?

(*) Maximization of the BGS entropy using *p*-moments leads to flexible power-type distributions (including the Pareto and Tsallis distributions for $q = 1$ and $q = 2$, respectively).

(*) *p*-moments are simple and, for $p = 0$, become identical to the ordinary moments.

(*) They exhibit similar properties with the ln$x$ function, and thus are suitable for positively skewed RVs; additionally, compared to $E(\ln x)$ they are always positive.

For interested: more on this subject can be learned from

(*) **http://arxiv.org/abs/cond-mat/0507414v1**
A.Rostovtsev, *On a geometric mean and power-law statistical distributions.*

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Generalized power function: $x_p^q = \ln(1 + p\, x^q)/p \quad \xrightarrow[p \to 0]{} x^q$

p – moments of order q : $m_q^p = E(X_p^q) = \frac{1}{p}\int_0^\infty \ln(1 + p\, x^q)\, f_X(x)\mathrm{d}x$

$\xrightarrow[p \to 0]{} m_q \equiv E(X^q)$ classical moments

8. Entropy maximization based on p-moments

The following table displays distributions (in terms of Lagrange multipliers λ_j) arising from the maximization of the BGS entropy and by imposing constraints, (a) classical moments m_q of various orders, (b) p-moments or various orders, and (c) combinations of moments or p-moments with the expectation of ln x. In all cases classical moments produce exponential-type distributions while p-moments produce power-type distributions.

Constraints	Distribution Name	Density function
m_1	Exponential	$f_X(x) = \exp(-\lambda_0 - \lambda_1 x)$
m_2	Half-Normal	$f_X(x) = \exp(-\lambda_0 - \lambda_1 x^2)$
m_1 and m_2	Normal	$f_X(x) = \exp(-\lambda_0 - \lambda_1 x - \lambda_2 x^2)$
m_q	Generalized Exponential	$f_X(x) = \exp(-\lambda_0 - \lambda_1 x^q)$
$E(\ln x)$ and m_1	Gamma	$f_X(x) = x^{-\lambda_1} \exp(-\lambda_0 - \lambda_2 x)$
$E(\ln x)$ and m_q	Generalized Gamma	$f_X(x) = x^{-\lambda_1} \exp(-\lambda_0 - \lambda_2 x^q)$
m_1^p	Pareto type II	$f_X(x) = \exp(-\lambda_0)(1 + px)^{-\lambda_1/p}$
m_2^p	Tsallis	$f_X(x) = \exp(-\lambda_0)(1 + px^2)^{-\lambda_1/p}$
m_1^p and m_2^p	Not named	$f_X(x) = \exp(-\lambda_0)\left[(1 + px)^{\lambda_1}(1 + px^2)^{\lambda_2}\right]^{-1/p}$
$E(\ln x)$ and m_1^p	Beta of the second kind	$f_X(x) = \exp(-\lambda_0)\, x^{-\lambda_1}(1 + px)^{-\lambda_2/p}$
$E(\ln x)$ and m_q^p	Generalized Beta of the second kind	$f_X(x) = \exp(-\lambda_0)\, x^{-\lambda_1}(1 + px^q)^{-\lambda_2/p}$

(*) Conclusions

From the examples presented here it should be

realized that the widely discussed origin of Tsallis

distribution as emerging fromTsallis entropy,

is by no means the only possibility.

It also arises from many nonthermal sources without

really resorting to Tsallis entropy.