Anisotropic Hydrodynamics and early stages of the Quark Gluon Plasma

Outline

- Second order viscous hydrodynamics
- Expansion around an anisotropic background
- Leading order in anisotropic hydrodynamics
- Extension of the model

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X Polish Workshop on Relativistic Heavy-Ion Collisions
Kielce, Poland December 13-15, 2013

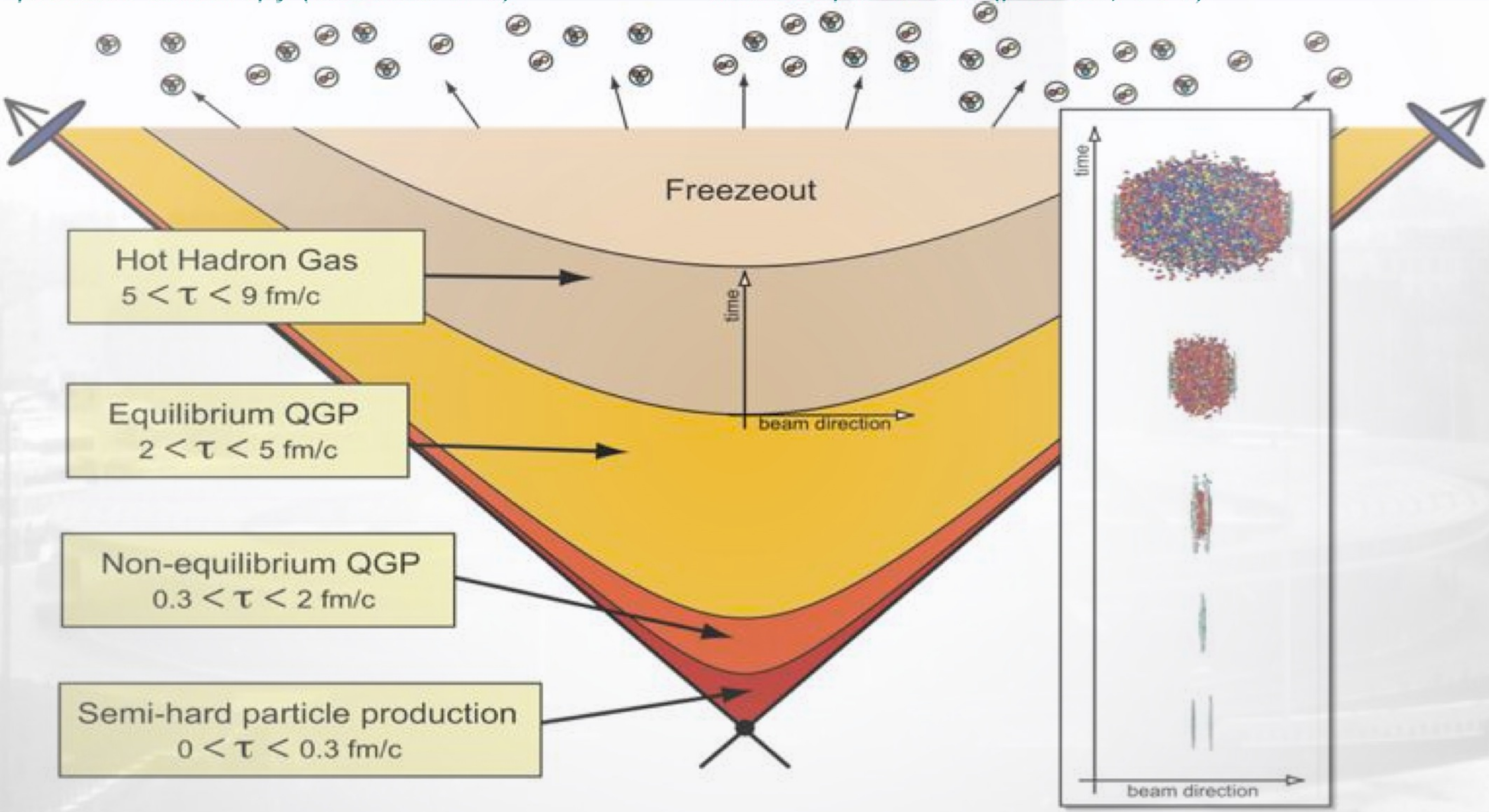
Motivations

Hydrodynamic modeling of heavy ion collisions (small viscosity)

Large gradients → *viscous corrections*

Strong longitudinal expansion, pressure anisotropy (also AdS/CFT)

Large momentum anisotropy from microscopic models (pQCCD, CGC)



From kinetic theory to hydrodynamics

Relativistic Boltzmann equation:

$$p^\mu \, \partial_\mu f(x,p) = \mathcal{C}[f]$$

First moment:

$$\int dP \, p^\mu p^\nu \, \partial_\mu f = \partial_\mu T^{\mu\nu} = \\ = \int dP \, p^\nu \mathcal{C}[f] \left[= 0 \right]$$

Hydrodynamics

Ansatz for the relativistic Boltzmann distribution

Perfect fluid:

$$f \simeq f_{\text{eq.}} = k \exp\left[-\frac{p \cdot U(x)}{T(x)}\right]$$

$$T^{\mu\nu} = k \int dP \, p^\mu p^\nu \exp\left[-\frac{p \cdot U}{T}\right] = (\varepsilon + P) U^\mu U^\nu - g^{\mu\nu} P$$

$$T_{\text{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Dissipative hydrodynamics

$$f = f_{\mathrm{eq.}} + \delta f \quad \Rightarrow \quad T^{\mu\nu} = T^{\mu\nu}_{\mathrm{eq.}} + \delta T^{\mu\nu}$$

$\delta f \Rightarrow \delta T^{\mu\nu}$ treated as, small, perturbations

Landau frame,
massless particles

$$\delta T^{\mu\nu} = \pi^{\mu\nu}$$

$$U^{\mu}\pi_{\mu\nu} = 0 \qquad g_{\mu\nu}\pi^{\mu\nu} = 0$$

Four equations, five more degrees of freedom!

Entropy current and entropy source

$$\mathcal{S}^\mu = \mathcal{S}^\mu_{\mathrm{eq.}} + \delta\mathcal{S}^\mu \simeq \frac{p}{T}U^\mu + \frac{1}{T}T^{\mu\nu}U_\nu - \frac{\tau_\pi}{4\eta T}\pi^{\alpha\beta}\pi_{\alpha\beta}$$

$$\partial_\mu \mathcal{S}^\mu \simeq \frac{1}{T}\pi^{\mu\nu}\left[\sigma_{\mu\nu} - \frac{\tau_\pi}{2\eta}\Delta^\alpha_\mu\Delta^\beta_\nu D\pi_{\alpha\beta} - \frac{1}{2}T\pi_{\mu\nu}\partial\cdot\left(\frac{\tau_\pi}{2\eta T}U\right)\right]$$

$$\tau_\pi\Delta^\alpha_\mu\Delta^\beta_\nu D\pi_{\alpha\beta} + \pi_{\mu\nu} = 2\eta\sigma_{\mu\nu} - \pi_{\mu\nu}T\eta\,\partial\cdot\left(\frac{\tau_\pi}{2\eta T}U\right)$$

Anisotropic hydrodynamics

Reorganization of the hydrodynamic expansion

$$f = f_{\mathrm{eq.}} + \delta f$$

around an anisotropic background instead of the local equilibrium

$$f = f_{\mathrm{aniso.}} + \delta \tilde{f}$$

“Romatschke-Strickland” form:

$$f_{\mathrm{aniso.}} = k \exp\left[-\frac{\sqrt{\left(p \cdot U(x) \right)^2 + \xi(x) \left(p \cdot Z(x) \right)^2}}{\Lambda(x)} \right]$$

0+1 dimensions

Boost invariant in the longitudinal direction, homogeneous in the transverse plane

$$T_{\text{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P_\perp & 0 & 0 \\ 0 & 0 & P_\perp & 0 \\ 0 & 0 & 0 & P_\parallel \end{pmatrix}$$

No negative pressure already at the leading order

$$\left(P_\perp, P_\parallel \right) = \int dP \left(\frac{1}{2} p_\perp^2, p_\parallel^2 \right) f_{\text{RS}} > 0$$

Positive by construction

While, in the Navier-Stokes limit

$$\frac{P_\parallel}{P_\perp} = \frac{P_{\text{eq.}} + \pi_{ZZ}}{P_{\text{eq.}} + \pi_{XX}} \simeq \frac{3T\tau - 16\bar{\eta}}{3T\tau + 8\bar{\eta}}$$

Exact solution! Test for viscous and anisotropic hydrodynamics

0+1 dimensions

Strong constraints due to symmetry ⇨ any scalar only depends on the proper time

$$\tau = \sqrt{t^2 - z^2}$$

The Boltzmann equation in relaxation time approximation reads

$$p^\mu \partial_\mu f = p_\mu U^\mu \frac{f_{eq.} - f}{\tau_{eq.}} \longrightarrow \frac{\partial f}{\partial \tau} = \frac{f_{eq.} - f}{\tau_{eq.}}$$

Comparison between

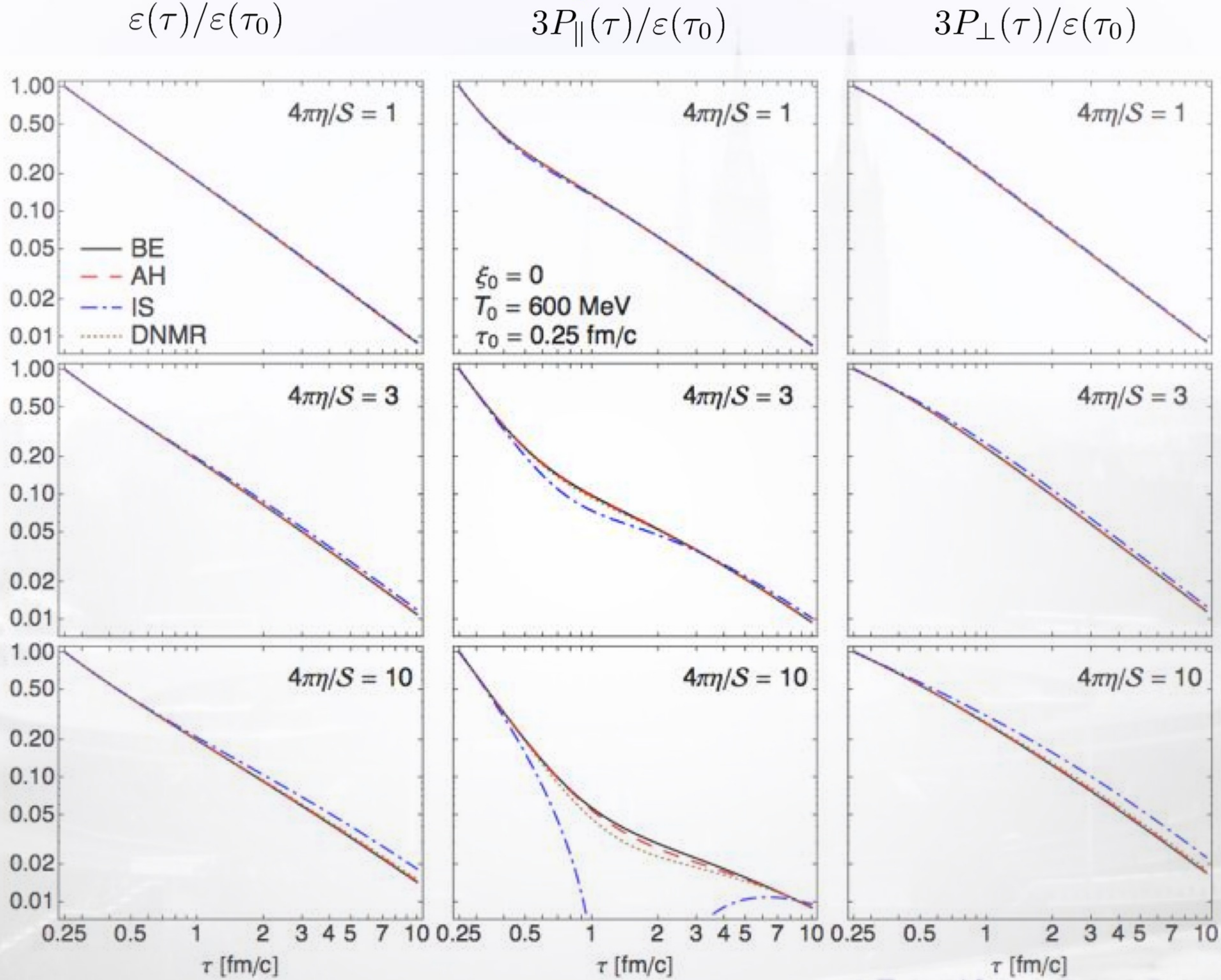
- Exact solution

$$f = \exp\left[-\int_{\tau_0}^{\tau} \frac{d\tau''}{\tau_{eq.}}\right] f_0 + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{eq.}} \exp\left[-\int_{\tau'}^{\tau} \frac{d\tau''}{\tau_{eq.}}\right] f_{eq.}$$

- Leading order in the anisotropic expansion (AH)
- Israel-Stewart (IS)
- Denicol, Niemi, Molnar, Rischke, Phys. Rev. D 85, 114047 (2012). (DNMR)

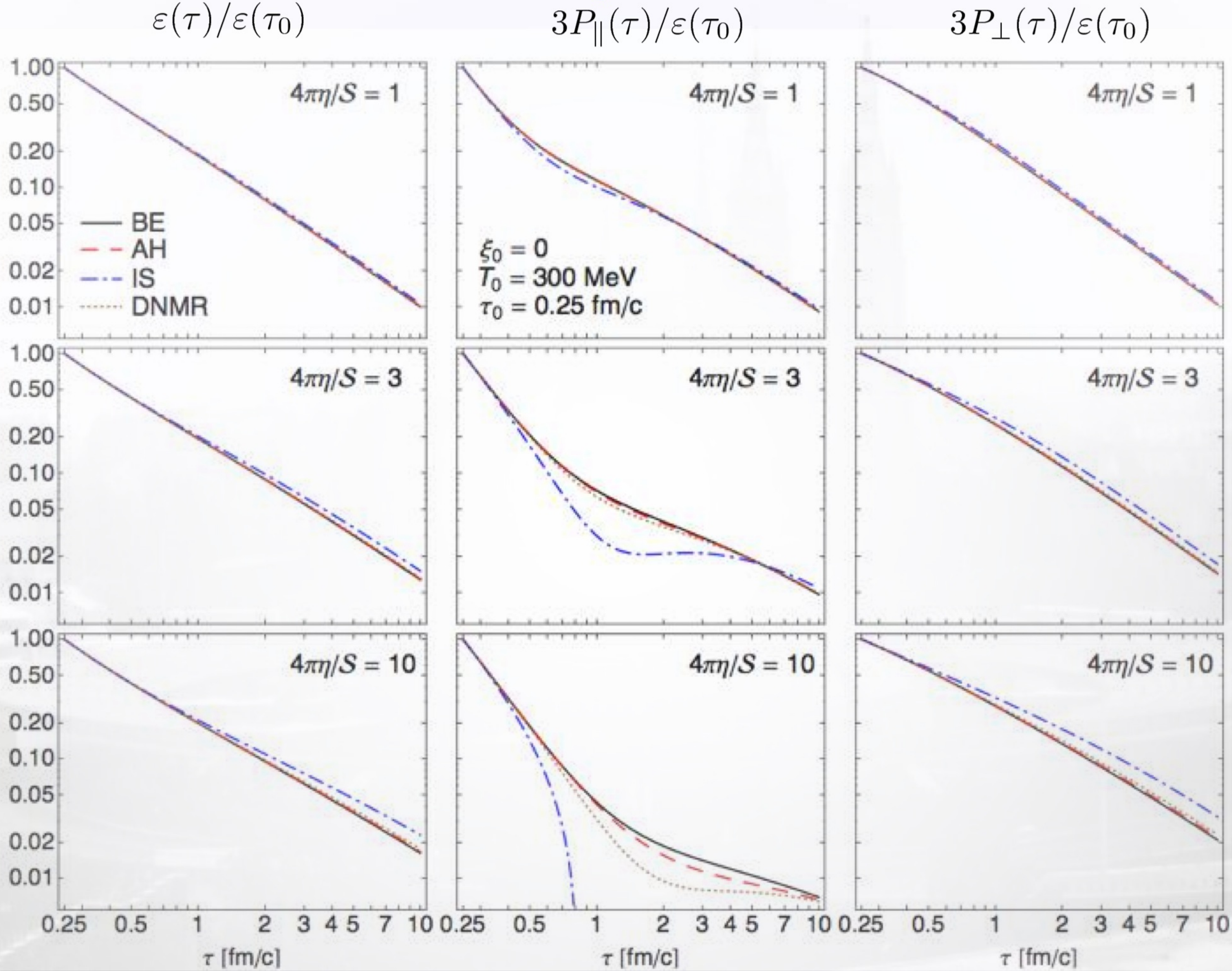
- **W Florkowski, R Ryblewski and M Strickland, Phys. Rev. C88, 024903 (2013)**

Some plots



$\varepsilon(\tau)/\varepsilon(\tau_0)$
$3P_{\parallel}(\tau)/\varepsilon(\tau_0)$
$3P_{\perp}(\tau)/\varepsilon(\tau_0)$
4πη/S = 1
4πη/S = 3
4πη/S = 10
BE
AH
IS
DNMR
ξ0 = 0
T0 = 600 MeV
τ0 = 0.25 fm/c
1.00
0.50
0.20
0.10
0.05
0.02
0.01
0.25 0.5 1 2 3 4 5 7 10
τ [fm/c]

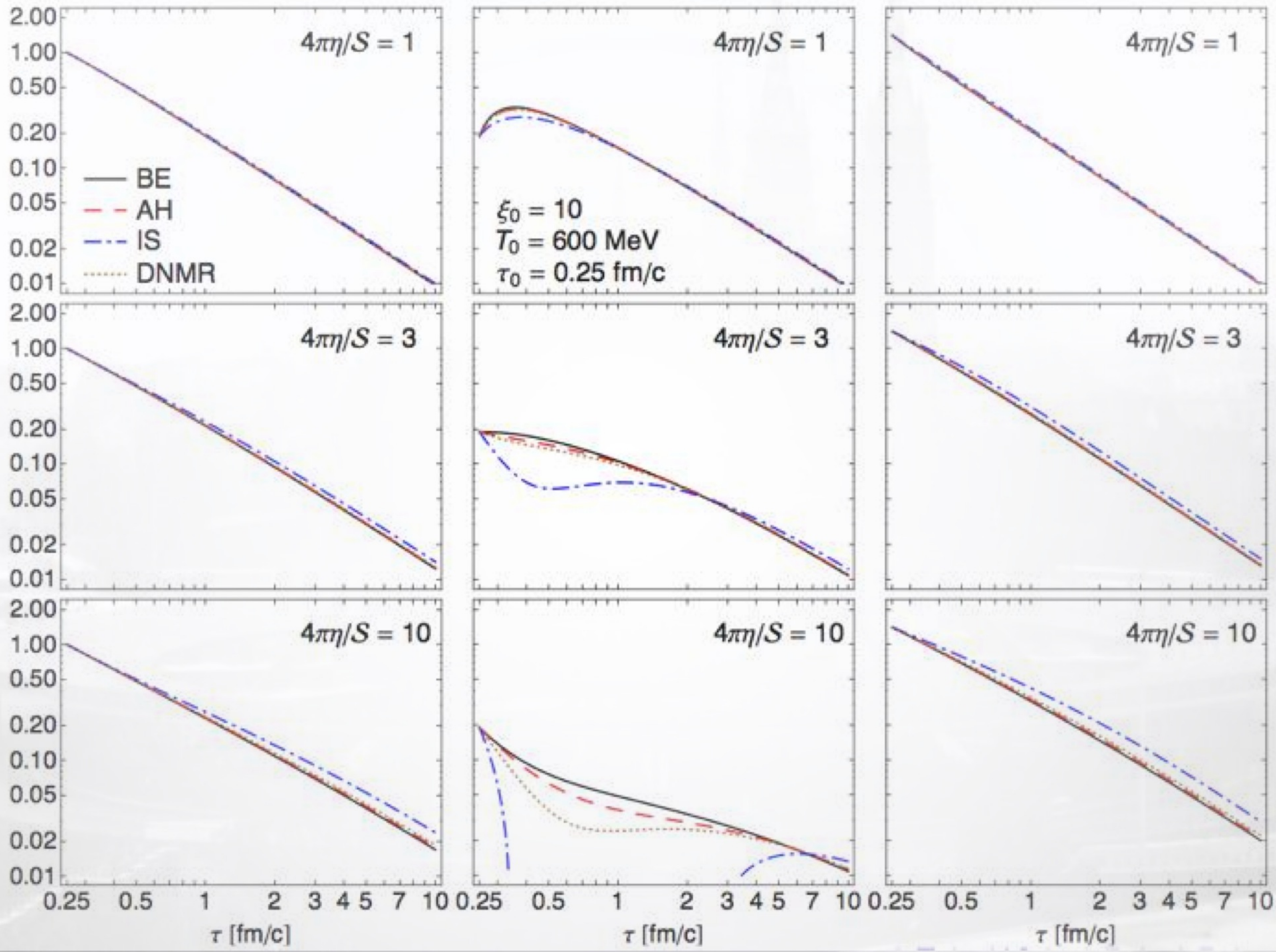
Some plots



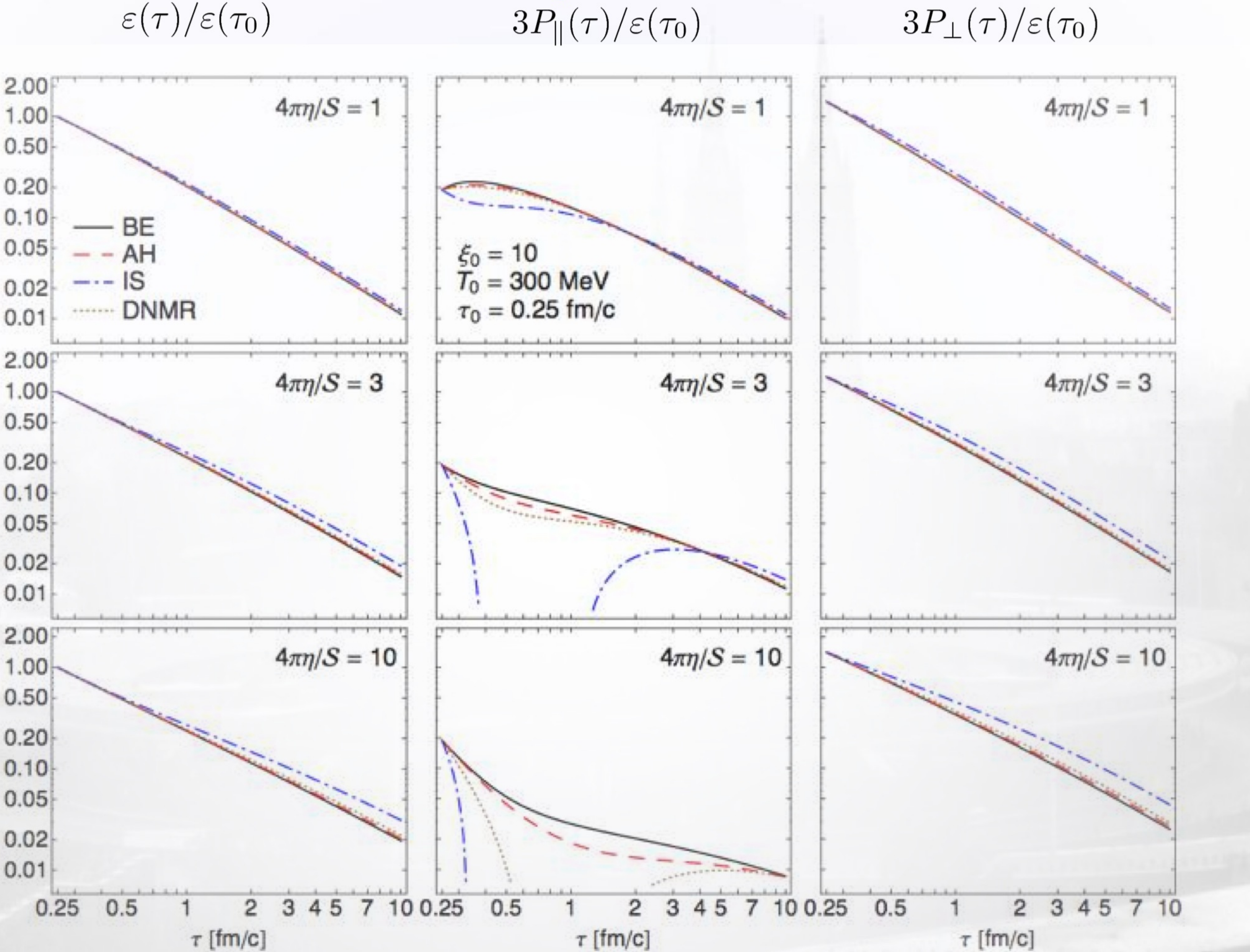
$\varepsilon(\tau)/\varepsilon(\tau_0)$
$3P_{\parallel}(\tau)/\varepsilon(\tau_0)$
$3P_{\perp}(\tau)/\varepsilon(\tau_0)$
4πη/S = 1
4πη/S = 3
4πη/S = 10
BE
AH
IS
DNMR
ξ0 = 0
T0 = 300 MeV
τ0 = 0.25 fm/c
1.00
0.50
0.20
0.10
0.05
0.02
0.01
0.25
0.5
1
2
3
4
5
7
10
τ [fm/c]

Some plots

$\varepsilon(\tau)/\varepsilon(\tau_0)$ $3P_{\|}(\tau)/\varepsilon(\tau_0)$ $3P_{\perp}(\tau)/\varepsilon(\tau_0)$



Some plots



$\varepsilon(\tau)/\varepsilon(\tau_0)$
$3P_{\|}(\tau)/\varepsilon(\tau_0)$
$3P_{\perp}(\tau)/\varepsilon(\tau_0)$
$4\pi\eta/S = 1$
$4\pi\eta/S = 3$
$4\pi\eta/S = 10$
BE
AH
IS
DNMR
$\xi_0 = 10$
$T_0 = 300$ MeV
$\tau_0 = 0.25$ fm/c
2.00
1.00
0.50
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1
2
3
4
5
7
10
τ [fm/c]

Radial flow

1+1 dimensions, boost invariance in the longitudinal direction, rotation invariance in the transverse plane

Four-velocity

$$U = \gamma \begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \cosh\theta_\perp \cosh\eta_{||} \\ \sinh\theta_\perp \cos\phi \\ \sinh\theta_\perp \sin\phi \\ \cosh\theta_\perp \sinh\eta_{||} \end{pmatrix}$$

Another unknown function

$$\tanh\theta_\perp(\tau, r) = \cosh\eta_{||}\sqrt{v_x^2 + v_y^2}$$

Orthonormal basis

$$X = \begin{pmatrix} \sinh\theta_\perp \cosh\eta_{||} \\ \cosh\theta_\perp \cos\phi \\ \cosh\theta_\perp \sin\phi \\ \sinh\theta_\perp \sinh\eta_{||} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 \\ -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \qquad Z = \begin{pmatrix} \sinh\eta_{||} \\ 0 \\ 0 \\ \cosh\eta_{||} \end{pmatrix}$$

- **W Florkowski and R Ryblewski, Phys. Rev. C 85, 044902 (2012)**

Radial flow

Only two of the four equations of the local energy momentum conservation are not trivial

Projecting the first moment of the Boltzmann equation $\partial_\mu T^{\mu\nu} = 0$

along the vectors of the basis $U^\mu \quad X^\mu \quad Y^\mu \quad Z^\mu$

results in two trivial equations and the projections along the four-velocity and radial direction

$$(\cosh\theta_\perp \partial_\tau + \sinh\theta_\perp \partial_r)\,\varepsilon + \varepsilon \left[\cosh\theta_\perp \left(\frac{1}{\tau} + \partial_r \theta_\perp\right) + \sinh\theta_\perp \left(\frac{1}{r} + \partial_\tau \theta_\perp\right)\right]$$
$$+P_\perp \left(\cosh\theta_\perp \partial_r \theta_\perp + \sinh\theta_\perp \partial_\tau \theta_\perp + \frac{\sinh\theta_\perp}{r}\right) + P_\parallel \frac{\cosh\theta_\perp}{\tau} = 0$$

$$(\sinh\theta_\perp \partial_\tau + \cosh\theta_\perp \partial_r)\,P_\perp + \varepsilon\,(\sinh\theta_\perp \partial_r \theta_\perp + \cosh\theta_\perp \partial_\tau \theta_\perp)$$
$$+P_\perp \left[\sinh\theta_\perp \left(\frac{1}{\tau} + \partial_r \theta_\perp\right) + \cosh\theta_\perp \left(\frac{1}{r} + \partial_\tau \theta_\perp\right) - \frac{\cosh\theta_\perp}{r}\right] - P_Z \frac{\sinh\theta_\perp}{\tau} = 0$$

We can use the zeroth moment of the Botzman equation for closing the system of equations, but...

No matching with second order viscous hydrodynamics

$$\sigma_{\mu\nu}X^\mu X^\nu = \frac{\cosh\theta_\perp}{3\tau} + \frac{\sinh\theta_\perp}{3r} - \frac{2}{3}\frac{\partial\theta_\perp}{\partial\tau}\sinh\theta_\perp - \frac{2}{3}\frac{\partial\theta_\perp}{\partial r}\cosh\theta_\perp$$

$$\sigma_{\mu\nu}Y^\mu Y^\nu = \frac{\cosh\theta_\perp}{3\tau} - \frac{2\sinh\theta_\perp}{3r} + \frac{1}{3}\frac{\partial\theta_\perp}{\partial\tau}\sinh\theta_\perp + \frac{1}{3}\frac{\partial\theta_\perp}{\partial r}\cosh\theta_\perp$$

The pressure corrections depend on the components of the shear stress tensor (proportional in the Navier-Stokes limit)

But using the Romatscchke-Strickland form for the leading order of the expansion, the pressure in the transverse plane is isotropic

Solutions?

- Anisotropic expansion for the early stages, then switch to viscous hydrodynamics?
- Higher order in the anisotropic expansion (M Strickland, U Heinz et al.)?

Improving the starting point of the expansion

For a conformal system

$$f_{\text{aniso.}} = k \exp\left[-\frac{\sqrt{(1+\xi_X)\left(p\cdot X\right)^2 + (1+\xi_Y)\left(p\cdot Y\right)^2 + (1+\xi_Z)\left(p\cdot Z\right)^2}}{\lambda(x)}\right]$$

$$\sum_I \xi_I = 0$$

Pressure not isotropic in the transverse plane

$$T_{\text{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P_X & 0 & 0 \\ 0 & 0 & P_Y & 0 \\ 0 & 0 & 0 & P_Z \end{pmatrix}$$

Still two trivial equations from the local four-momentum conservation and four unknown scalar function:$\theta_\perp, \lambda$ and two independent anisotropy parameter

- L Tinti and W Florkowski , coming soon ...

The remaining equations

Looking at the second moment of the Boltzmann equation, in relaxation time approximation, to close the system.

$$\partial_\lambda \Theta^{\lambda\mu\nu} = \frac{1}{\tau_{\rm eq}} \left(U_\lambda \Theta^{\lambda\mu\nu}_{\rm eq} - U_\lambda \Theta^{\lambda\mu\nu} \right)$$

$$\Theta^{\lambda\mu\nu} = \int dP \, p^\lambda p^\mu p^\nu f \qquad \Theta^{\lambda\mu\nu}_{\rm eq} = \int dP \, p^\lambda p^\mu p^\nu f_{\rm eq.}$$

Many of the projections of these equations along the vectors of the basis are trivial, but there is a useful combination of the remaining ones

$$\frac{D\xi_I}{1+\xi_I} + 2\sigma_I + \frac{\xi_I}{\tau_{\rm eq.}} \left(\frac{T}{\lambda}\right)^5 \sqrt{\prod_J (1+\xi_J)} - \frac{1}{3} \sum_J \frac{D\xi_J}{1+\xi_J} = 0$$

$$\sigma^{\mu\nu} = \sum_I \sigma_I I^\mu I^\nu$$

Only two of the three equations are independent, if two of them are fulfilled all of them are

Small anisotropy limit

$$\pi^{\mu\nu} = \sum_I \pi_I I^\mu I^\nu$$

Close to local equilibrium

$$\pi_I \simeq -\frac{4}{15}\,\varepsilon\,\xi_I \quad \Rightarrow \quad \xi_I \simeq -\frac{15}{4}\frac{\pi_I}{\varepsilon}$$

Using the Landau Matching

$$D\xi_I + 2\sigma_I + \frac{\xi_I}{\tau_{\text{eq.}}} \simeq 0$$

Multiplying by

$$-\frac{4}{15}\,\varepsilon$$

And the energy conservation equation

$$\tau_{\text{eq.}} D\pi_I + \pi_I = 2\left(\frac{4}{15}\,\varepsilon\,\tau_{\text{eq.}}\right)\sigma_I - \pi_I\left[\frac{4}{15}\,\varepsilon\tau_{\text{eq.}} T\partial\cdot\left(\frac{15}{8\varepsilon T}U\right)\right]$$

Consistent with Israel-Stewart

$$\tau_\pi\, D\pi_I + \pi_I = 2\eta\sigma_I - \pi_I\left[\eta T\partial_\lambda\left(\frac{\tau_\pi}{2\eta T}U^\lambda\right)\right]$$

provided

$$\tau_\pi = \tau_{\text{eq.}} \quad \eta = \frac{4}{15}\,\varepsilon\tau_{\text{eq.}} \Rightarrow \tau_\pi = \frac{5\bar{\eta}}{T}$$

Summary & outlook

- Anisotropic hydrodynamics is a reorganization of the hydrodynamic expansion, around an anisotropic distribution.
- Pressure anisotropies already at the leading order. Very good approximation in the exactly solvable 0+1 case
- Generalized ansatz for the leading order. Well defined in the early stages and consistent with second order viscous hydrodynamics close to local equilibrium in presence of radial flow.
- NEXT: extension to 2+1 or 3+1 dynamics? How much “physics” can we include in the leading order? Higher order in the, anisotropic, expansion?