Holography and hydrodynamics

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Abstract

Holography has made it possible to study the emergence of a relativistic hydrodynamic description from a highly non-equilibrium initial state of the strongly coupled N=4 supersymmetric Yang-Mills theory. Numerical studies suggest that a “hydrodynamic attractor” (possibly coresponding to a resummation of the hydrodynamic gradient expansion) exists at very early times. The Mueller-Israel-Stuart theory of second order hydrodynamics can be interpreted as a simple phenomenlogical model of such an attractor.

Plan

- How we use holography
- New numerical simulations of boost invariant flow
- A phenomenological model of the emergence of hydrodynamics

Holography

A quantum theory of gravity can be equivalently described in terms of a quantum field theory in one spacial dimension less.

Prime example: string theory on AdS_5 is the same physical theory as $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in $d = 1 + 3$ dimensions with gauge group $SU(N_c)$.

In the 't Hooft limit ($N_c \gg 1, \lambda \equiv g^2 N_c \gg 1$) string theory can be approximated by Einstein gravity.

Classical solutions of Einstein equations contain all the information about quantum $\mathcal{N} = 4$ SYM theory in this limit.

The expectation value of the energy momentum tensor describing boost-invariant flow of a conformal fluid takes the form

$$T_{\mu\nu} = \mathrm{diag}(\epsilon, p_{||}, p_{\perp}, p_{\perp})$$

where

$$\begin{aligned} p_{||} &= -\epsilon(\tau) - \tau\epsilon'(\tau) \\ p_{\perp} &= \epsilon(\tau) + \frac{1}{2}\tau\epsilon'(\tau) \end{aligned}$$

Here the Minkowski metric is expressed in proper time – rapidity coordinates, so

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_1^2 + dx_2^2$$

Holography leads to the formula

$$\epsilon(\tau) = -\frac{3N_c^2}{8\pi} a_4(\tau)$$

where the 5-dimensional metric

$$ds^2 = -\frac{1}{z^2}\left(1 + a_4(\tau)z^4 + \ldots\right) d\tau^2 + \ldots$$

is a solution to Einstein equations in five dimensions.

Simulations of BIF

Numerical solutions describing the non-equilibrium (pre-hydro) regime have been found by various groups (Chesler, Yaffe; Heller, Janik, Witaszczyk).

Recently, a new numerical approach was developed (Heller, MS) which makes it possible to

- study large numbers of initial conditions
- set initial conditions at arbitrarily small τ

This opens the door to searching for **universal behaviour** very early on (even before 1st or 2nd orders in the gradient expansion give a good account of numerical data).

How can one recognize that hydrodynamics provides a good description at a given time?

The hydrodynamic expansion (**up to third order** in gradients) is given by

$$\epsilon_H(\tau) = \frac{3N_c^2\pi^2}{8}\frac{\Lambda^4}{(\Lambda\tau)^{4/3}}\left\{1 - \frac{2}{3\pi}\cdot\frac{1}{(\Lambda\tau)^{2/3}} + \frac{1+2\log(2)}{18\pi^2}\cdot\frac{1}{(\Lambda\tau)^{4/3}} + \right.$$
$$\left. + \frac{-3+2\pi^2+24\log(2)-24\log^2(2)}{486\pi^3}\cdot\frac{1}{(\Lambda\tau)^2} + \ldots\right\}$$

One needs to compare this to $\epsilon(\tau)$ obtained from numerics (which will also contain contributions from non-hydrodynamic modes).

In the case of Bjorken flow the **equations of hydrodynamics** take the form:

$$\frac{\tau}{w}\frac{d}{d\tau}w = f_H(w)$$

where $w = \tau\epsilon(\tau)^{1/4}$. The function $f_H(w)$ can be computed order by order in the gradient expansion.

For $\mathcal{N} = 4$ SYM one finds

$$f_H(w) = \frac{2}{3} + \frac{1}{9\pi w} + \frac{1-2\log(2)}{27\pi^2 w^2} + \\ + \frac{15 - 2\pi^2 - 45\log(2) + 24\log^2(2)}{972\pi^3 w^3} + \ldots$$

By calculating

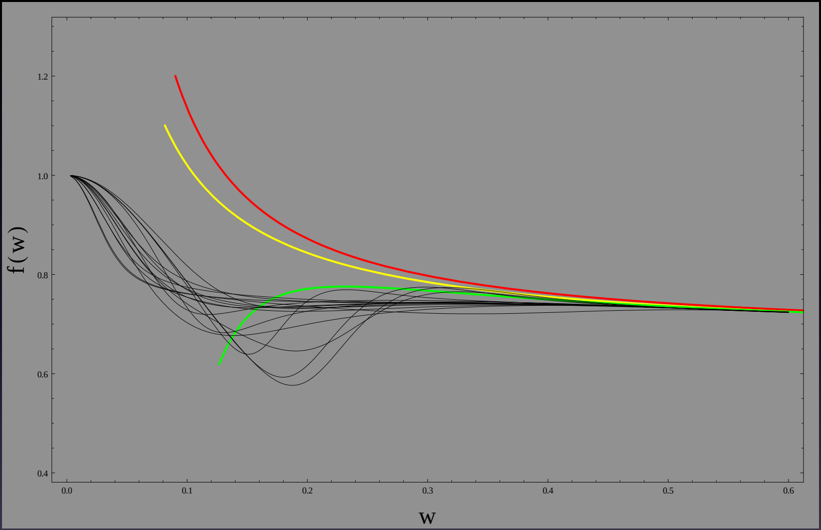
$$f(w) = \frac{\tau}{w}\frac{d}{d\tau}w$$

for a numerically computed $\epsilon(\tau)$ one can check if and when hydrodynamic behaviour emerges.

This provides a practical way of recognizing the transition to hydrodynamics starting from a far from equilibrium initial state.

Plotting the numerically computed $f(w)$ for randomly generated initial conditions shows

- diverse behaviour early on
- convergence to hydrodynamics at late times



yellow, red, green – 1st, 2nd, 3rd order gradient expansion

black – numerics.

These results suggest that there may exist a **universal attractor** well before 1st/2nd order hydrodynamics becomes valid ("resummed hydrodynamics").

Early speculations about all order hydrodynamics include the work of Shuryak and Lublinsky (2009). Also the anisotropic hydrodynamics approach of Florkowski and Ryblewski (2010) has a similar flavour.

It may be possible to determine this attractor by **resummation** of the gradient expansion - first steps in this direction were made by Janik, Heller, Witaszczyk (2013).

It is however very likely that at least some **non-hydrodynamic modes** will have to be taken into account in any phenomenological model. On the gravity side they correspond to low lying quasinormal modes.

Müller-Israel-Stewart theory

The MIS theory of “second order hydrodynamics” is a standard framework for simulating quark gluon plasma evolution.

This theory

- involves **non-hydrodynamic modes** as well as hydrodynamic modes;
- contains hydrodynamics as a **late-time attractor**
- includes hydrodynamic modes **to all orders** in the gradient expansion;

Thus, it provides a phenomenological model of the transition to hydrodynamics.

(Heller, Janik, MS, Witaszczyk, in preparation)

If the energy momentum tensor is

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + p P^{\mu\nu} + \Pi^{\mu\nu}$$

then in conformal hydrodynamics up to 2nd order

$$\begin{aligned}\Pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} + \eta\tau_\Pi\left[{}^{<}D\sigma^{\mu\nu>} + \frac{1}{3}\sigma^{\mu\nu}(\nabla\cdot u)\right] + \\ &+ \lambda_1\sigma^{<\mu}{}_{\lambda}\sigma^{\nu>\lambda} + \lambda_2\sigma^{<\mu}{}_{\lambda}\Omega^{\nu>\lambda} + \lambda_3\Omega^{<\mu}{}_{\lambda}\Omega^{\nu>\lambda}\,.\end{aligned}$$

In MIS one treats $\Pi^{\mu\nu}$ as an **additional dynamical variable** determined by the evolution equation

$$\begin{aligned}\Pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} - \tau_\Pi\left[{}^{<}D\Pi^{\mu\nu>} + \frac{d}{d-1}\Pi^{\mu\nu}(\nabla\cdot u)\right] \\ &+ \frac{\lambda_1}{\eta^2}\Pi^{<\mu}{}_{\lambda}\Pi^{\nu>\lambda} - \frac{\lambda_2}{\eta}\Pi^{<\mu}{}_{\lambda}\Omega^{\nu>\lambda} + \lambda_3\Omega^{<\mu}{}_{\lambda}\Omega^{\nu>\lambda}\,.\end{aligned}$$

(Baier, Romatschke, Son, Starinets, Stephanov 2008).

For BIF the MIS equations are

$$
\begin{aligned}
\tau\epsilon'(\tau) &= -\frac{4\epsilon(\tau)}{3} + \phi(\tau) \\
\tau_{\Pi}(\tau)\phi'(\tau) &= \frac{4\eta(\tau)}{3\tau} - \frac{\lambda_1(\tau)\phi(\tau)^2}{2\eta(\tau)^2} - \frac{4\tau_{\Pi}(\tau)\phi(\tau)}{3\tau} - \phi(\tau)
\end{aligned}
$$

where $\phi \equiv -\Pi^y_y$.

Conformal symmetry requires that

$$
\begin{aligned}
\tau_{\Pi}(\tau) &= \frac{C_{\tau\Pi}}{\epsilon(\tau)^{1/4}} \\
\lambda_1(\tau) &= C_{\lambda_1}\frac{\eta(\tau)}{\epsilon(\tau)^{1/4}} \\
\eta(\tau) &= C_\eta \epsilon(\tau)^{3/4}
\end{aligned}
$$

where $C_\eta, C_{\tau\Pi}, C_{\lambda_1}$ are constants (whose values for strongly coupled $\mathcal{N} = 4$ SYM are known from holography).

By a change of variables in the MIS equations one can find an equation for the function f(w) in this theory

$$
\begin{aligned}
wC_{\tau\Pi} f(w) f'(w) \quad &+ \quad f(w)^2 \left(4C_{\tau\Pi} + \frac{3wC_{\lambda_1}}{2C_\eta} \right) \\
&+ \quad f(w) \left(-\frac{16C_{\tau\Pi}}{3} - \frac{2wC_{\lambda_1}}{C_\eta} + w \right) \\
&- \quad \frac{4C_\eta}{9} + \frac{16C_{\tau\Pi}}{9} - \frac{2w\left(C_\eta - C_{\lambda_1}\right)}{3C_\eta} = 0
\end{aligned}
$$

Hydrodynamics should appear as a **late-time attractor** in this equation. This particular solution will be denoted by f_H.

Hydrodynamics of finite order is obtained as a solution expanded **in powers of $1/w$**.

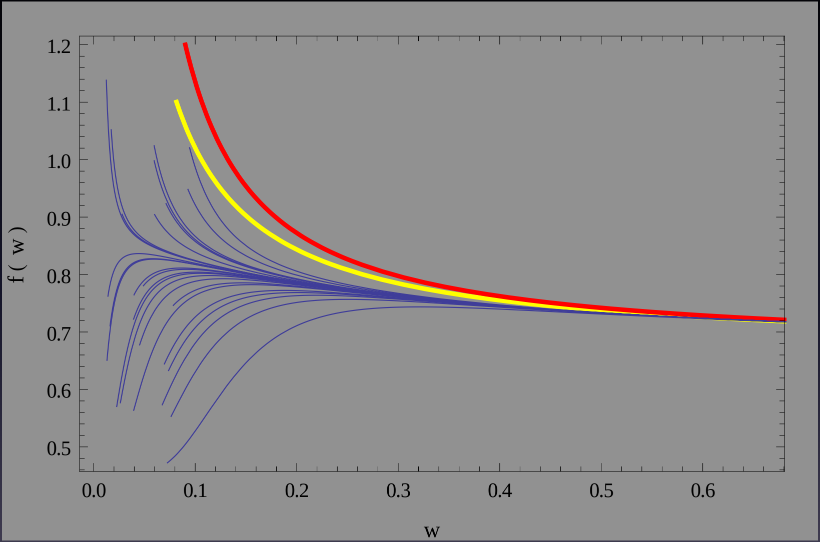
Using the values of transport coefficients for $\mathcal{N} = 4$ SYM obtained earlier via AdS/CFT one finds

$$f_H(w) = \frac{2}{3} + \frac{1}{9\pi w} + \frac{1 - \log(2)}{27\pi^2 w^2} + \frac{-2 + 4\log^2(2) - 5\log(2)}{162\pi^3 w^3} + \dots$$

The terms corresponding to hydrodynamics up to 2nd order match, while the **third order term is different**.

By looking at perturbations one can check that this is a **stable attractor** at linear level.

One can see the attractor by inspection:

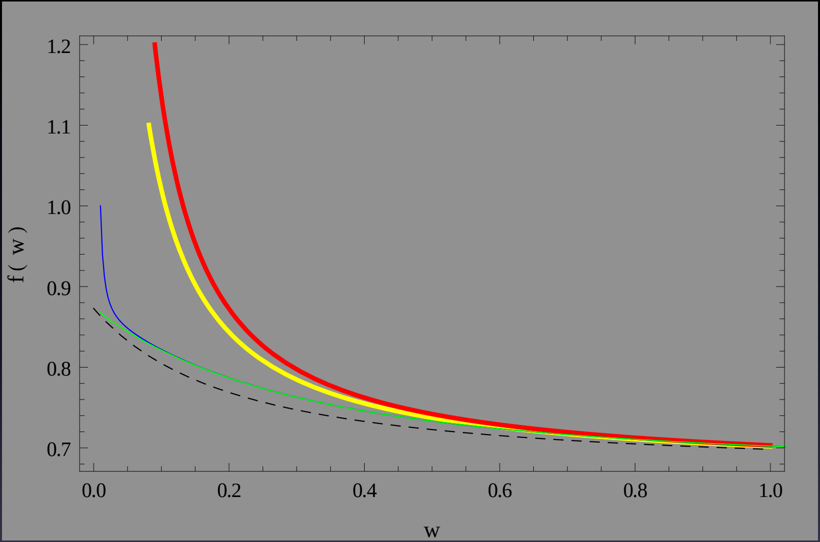


Initial states may involve nonhydro modes in varying degrees.
These modes decay “exponentially” with a scale set by $C_{\tau\Pi}$.

The hydrodynamic attractor can be determined numerically by setting appropriate initial conditions at small w.

An analytic, approximate formula can also be found by iteration: at leading order one finds

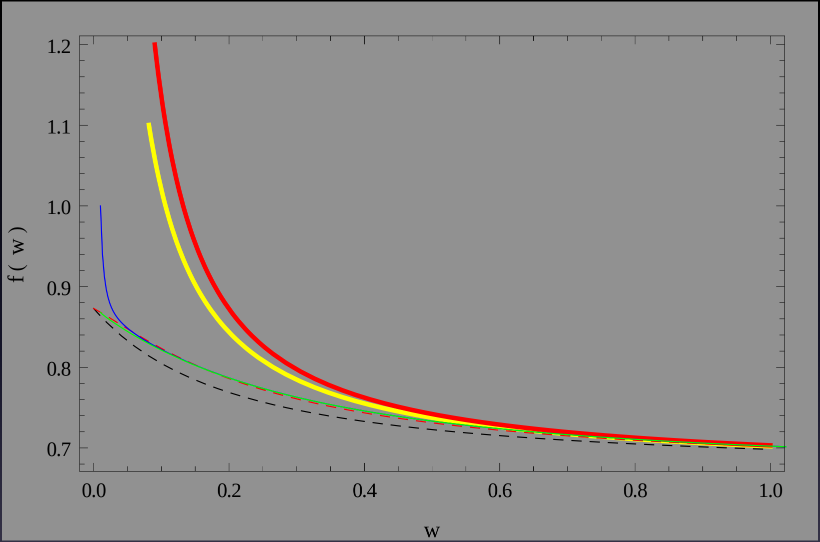
$$f(w) \approx \frac{\sqrt{9\pi^2 w^2 + 12\pi w + 16 - 8\log(2)} + 9\pi w + 16 - 8\log(2)}{6(3\pi w + 4 - 2\log(2))}$$



yellow, red – 1st, 2nd order hydro

green – hydro attractor; blue – MIS

dashed black – approximate formula



yellow, red – 1st, 2nd order hydro

green – hydro attractor; blue – MIS

dashed black, red – 1st, 2nd approximation.

Conclusions

- Holography provides an inspiring alternative to kinetic theory as a playground to study hydrodynamics
- "All order hydrodynamics" may be viewed as a late time attractor in a MIS type setting
- For very early times it may be possible to obtain an effective description involving both hydrodynamic and non-hydrodynamic modes which captures more features of Yang-Mills plasma dynamics than MIS